Fusion of the q-Vertex Operators and its Application to Solvable Vertex Models

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Abstract

We diagonalize the transfer matrix of the inhomogeneous vertex models of the 6-vertex type in the anti-ferroelectric regime intoducing new types of q-vertex operators. The special cases of those models were used to diagonalize the s-d exchange model[21, 1, 6]. New vertex operators are constructed from the level one vertex operators by the fusion procedure and have the description by bosons. In order to clarify the particle structure we estabish new isomorphisms of crystals. The results are very simple and figure out representation theoretically the ground state degenerations.

1 Introduction

In [2] the anti-ferroelectric XXZ hamiltonian, or equivalently, the transfer matrix of the 6-vertex model has been diagonalized directly in the thermodynamic limit based on the quantum affine symmetry. The method is powerful enough, on the one hand, to give the integral formulas for correlation functions and form factors, on the other hand, to determine the physical space as a representation of a quantum affine algebra.

Similar approach is possible for several two dimensional lattice models such as ABF model[11, 4]. Among them a direct generalization of the 6-vertex model is the vertex models associated with the perfect representations of any level[15, 16]. Although there are technical problems of bosonozation in the case of higher levels, at least the strategy is clear and everything we need is in our hands.

In this paper I want to add one more class of vertex models which can be solved by a similar method and are not contained in the class of directly generalized models said above. The vertex models which we study here is the inhomogeneous vertex models of 6-vertex type with the inhomogeneities being in the spins. Namely, on the infinite regular square lattice, with

each horizontal and vertical lines except a finite number of vertical lines $l_1, \cdots, l_n$, we associate the vector space $\mathbf{C}^2$. With $l_1, \cdots, l_n$ we associate $\mathbf{C}^{s_1+1}, \cdots, \mathbf{C}^{s_n+1}$ for arbitrary non-negative integers $s_1, \cdots, s_n$. To each vertex the Boltzmann weight is defined by the corresponding trigonometric R-matrix acting on $\mathbf{C}^2 \otimes \mathbf{C}^2$ or $\mathbf{C}^2 \otimes \mathbf{C}^{s_j}$. The rational limits of those models with $n = 1$ had been used to diagonalize the s-d exchange models (Kondo problem)[1, 21, 6, 20].

The central object in the symmetry approach is the q-vertex operator which was introduced by Frenkel-Reshetikhin[5]. In the case of the 6-vertex model the q-vertex operator makes it possible to identify the infinite tensor product $(\mathbf{C}^2)^{\otimes \mathbf{Z}_{\geq 1}}$ with the irreducible representation $V(\Lambda_i)$ of $U_q(\hat{s}l_2)$. Using this identification, the transfer matrix, the creation-annihilation operators, correlation functions and form factors are all described in terms of q-vertex operators.

Similarly, in our case, everything is described by q-vertex operators. But here appeares a new phenomenon, the degeneration of the ground states. To take this effect into consideration is crucial in the theory. To treate this situation correctly what we must do is to introduce new kinds of q-vertex operators. Those new operators can be considered as a mixture of type I and type II vertex operators in the terminology of [2]. Naturally they can be obtained by a fusion procedure from level one vertex operators. In particular new operators have the description by free fields. Hence physical quantities of our models can be written down in the form of integral formulas. We study these formulas in the next paper.

Let us describe our story more precicely. The total quantum space which is acted by the transfer matrix is

$$\oplus_{i,j=0,1} V(\Lambda_i) \otimes V_{s_n} \otimes \cdots \otimes V_{s_1} \otimes V(\Lambda_j)^{*a}, \tag{1}$$

where $V_s \simeq \mathbf{C}^{s+1}$ and considered as the representation of $U_q'(\hat{s}l_2)$. In order to give the description of the correlation function or form factors we must know the structure of eigenstates of the transfer matrix. The insight comes, as in the case of the XXZ-model[2, 7], from the decomposition of crystals

$$B(\Lambda_i) \otimes B_{s_1} \otimes \cdots \otimes B_{s_k} \otimes B(\Lambda_j)^*.$$

The result is surprisingly simple (see Corollary 1). Consequently we find that the physical space of our models can be written as

$$\mathbf{C}^{s_n} \otimes \cdots \otimes \mathbf{C}^{s_1} \otimes \Big[\oplus_{m=0}^{\infty} \int_{|z_1|=1} \cdots \int_{|z_m|=1} (\mathbf{C}^2)^{\otimes m} \Big]_{sym},$$

where sym is some symmetrization. In this tensor product the last term which is described by a bracket is the physical space of the XXZ model. On the other hand former tensor component describes the ground state degeneration. In the case $n = 1$ the dimension of the degeneracy of the ground states coinsides with the results of Fateev-Wiegman[6] in the rational limits. This picture of the structure of the space of states suggests that it is natural to consider the space

$$\oplus_{i,j=0,1} V_{s_n-1} \otimes \cdots \otimes V_{s_1-1} \otimes V(\Lambda_i) \otimes V(\Lambda_j)^{*a}. \tag{2}$$

The relation between two spaces 1 and 2 is given by the vertex operators

$$_{V_{s-1}}\Phi^{V_s}(z) : (V_{s-1})_z \otimes V(\Lambda_i) \longrightarrow V(\Lambda_{i+1}) \otimes (V_s)_z, \tag{3}$$

$$^{V_{s-1}}\Phi_{V_s}(z) : V(\Lambda_i) \otimes (V_s)_z \longrightarrow (V_{s-1})_z \otimes V(\Lambda_{i+1}), \tag{4}$$

which are parts of the newly introduced operators. On the space 2 descriptions of the model and physical quantities take very simple forms. For example the transfer matrix is, up to a scalar multiple, equal to $1 \otimes T_{XXZ}(z)$, where $T_{XXZ}(z)$ is the transfer matrix of the 6-vertex model. The peculiarity of our model comes from the definition of the local operators which are defined using vertex operators 3 and 4.

The present paper is organized in the following manner. In section 2 we review necessary preliminaries and notations. In section 3 we establish a new type of isomorphisms of crystals which is considered to be a generalization of the path realization of the crystals with highest weights. Applying this isomorphism we determine the decomposition of the crystal mentioned above. In section 4 we introduce a new vertex operators and prove their existence. The existence theorem explains why the spectral parameters in the right hand side and in the left hand side of 3, 4 must coinside. This fact is important to treat the ground state degeneration. The fusion construction of the representations and R-matrices are briefly reviewed in section 5. In section 6 the fusion procedure is carried out for level one vertex operators and construct new vertex operators. The well definedness of the fusion proceadure is the main result here. We determine the commutation relations of newly defined vertex operators using the fusion construction in section 7. In section 8 we propose the mathematical settings for our models. In appendix 1 the integral formulas for the highest-highest matrix element of the composition of type I and type II vertex operators are given. In appendix 2 ,3 the description of level one vertex operators in terms of bosons and their OPEs are given. These are used to derive the integral formulas in appendix 1.

2 Notations and preliminaries

2.1 Definition of quantized envelopping algebra

Let us recall the definition of $U_q(\hat{sl}_2)$ and fix several notations related to it. Let $P = \mathbf{Z}\Lambda_0 \oplus \mathbf{Z}\Lambda_1 \oplus \mathbf{Z}\delta$, $P^* = \mathbf{Z}h_0 \oplus \mathbf{Z}h_1 \oplus \mathbf{Z}d$ be the weight and the dual weight lattice of $\hat{sl}_2$ with the pairing $< \Lambda_i, h_j >= \delta_{ij}$, $< \Lambda_i, d >= 0$, $< \delta, h_i >= 0$, $< \delta, d >= 1$. Set $\alpha_1 = 2\Lambda_1 - 2\Lambda_0$, $\alpha_0 = \delta - \alpha_1$, $\rho = \Lambda_0 + \Lambda_1$. The symmetric bilinear form on P normalized as $(\alpha_i, \alpha_i) = 2$ is given by $(\Lambda_i, \Lambda_j) = \frac{\delta_{1i}\delta_{1j}}{2}$, $(\Lambda_i, \delta) = 1$, $(\delta, \delta) = 0$. Through $(,)$ we consider P^* as a subset of P so that $2\rho = h_1 + 4d$. Let us set $F = \mathbf{Q}(q)$ with q being the complex number transcendent over the rational number field $\mathbf{Q}$. In section 8, we assume that the q is real and $-1 < q < 0$.

The algebra $U_q(\hat{sl}_2)$ is the F-algebra generated by $e_i, f_i, (i = 0, 1), q^h (h \in P^*)$ with the defining relations

$$q^0 = 1, \quad q^{h_1} q^{h_2} = q^{h_1 + h_2}, \quad q^h e_i q^{-h} = q^{<h,\alpha_i>} e_i, \quad q^h f_i q^{-h} = q^{-<h,\alpha_i>} f_i,$$
$$[e_i, f_j] = \frac{t_i - t_i^{-1}}{q - q^{-1}}, \quad \sum_{m=0}^{3} (-1)^m x_i^{(m)} x_j x_i^{(3-m)} = 0 \quad (i \neq j) \text{ for } x = e, f$$

where we set $t_i = q^{h_i}$ and

$$x_i^{(m)} = \frac{x_i^m}{[m]!} \quad (x = e, f), \quad [m] = \frac{q^m - q^{-m}}{q - q^{-1}}, \quad [m]! = \prod_{k=1}^{m} [k], \quad \begin{bmatrix} n \\ j \end{bmatrix} = \frac{[n]!}{[j]![n-j]!}.$$

We denote by $U' = U_q'(\hat{s}l_2)$ the subalgebra of $U_q(\hat{s}l_2)$ generated by $e_i,\ f_i,\ t_i\ (i=0,1)$.

2.2 Hopf algebra structure

In this paper we use the following coproduct and the anti-pode,

$$\Delta(e_i) = e_i \otimes 1 + t_i \otimes e_i, \quad \Delta(f_i) = f_i \otimes t_i^{-1} + 1 \otimes f_i, \quad \Delta(q^h) = q^h \otimes q^h,$$
$$a(e_i) = -t_i^{-1} e_i, \quad a(f_i) = -f_i t_i, \quad a(q^h) = q^{-h}.$$

2.3 Finite dimensional module

The $U_q'(\hat{s}l_2)$ module $(V_n)_z = \oplus_{j=0}^n F[z,z^{-1}] v_j^{(n)}$ is defined as

$$f_1 v_j^{(n)} = [n-j] v_{j+1}^{(n)}, \quad e_1 v_j^{(n)} = [j] v_{j-1}^{(n)}, \quad t_1 v_j^{(n)} = q^{n-2j} v_{j+1}^{(n)},$$
$$f_0 = z^{-1} e_1, \quad e_0 = z f_1, \quad t_0 = t_1^{-1},$$

where z is a complex number. In particular $v_j^{(n)} = \begin{bmatrix} n \\ j \end{bmatrix}^{-1} \frac{v_0^{(n)}}{[j]!}$. In the following sections, for the sake of simplicity, we only write F instead of $F[z,z^{-1}]$ as far as no confusion occurs in every situation.

2.4 Dual module

For a left $U_q'(\hat{s}l_2)$-module M, we define the left module $M^{*a^{\pm 1}}$ by

$$M^{*a^{\pm 1}} = \mathrm{Hom}(M,F) \text{ as a linear space,}$$
$$< xw, v > = < w, a^{\pm 1}(x) v > \ \text{ for } w \in M^{*a^{\pm 1}},\ v \in M \text{ and } x \in U_q'(\hat{s}l_2).$$

Here the linear dual of an integrable module with finite dimensional weight spaces should be considered to be the restricted dual. By definition M, $M^{*a*a^{-1}}$ and $M^{*a^{-1}*a}$ are canonically isomorphic. For these dual modules the following properties hold,

$$\mathrm{Hom}_{U_q'(\hat{s}l_2)}(M_1 \otimes M_2, M_3) \simeq \mathrm{Hom}_{U_q'(\hat{s}l_2)}(M_1, M_3 \otimes M_2^{*a}),$$
$$\mathrm{Hom}_{U_q'(\hat{s}l_2)}(M_1 \otimes M_2, M_3) \simeq \mathrm{Hom}_{U_q'(\hat{s}l_2)}(M_2, M_1^{*a^{-1}} \otimes M_3),$$

where $\mathrm{Hom}_{U_q'(\hat{s}l_2)}(M_1, M_2)$ is the vector space of $U_q'(\hat{s}l_2)$ linear homomorphisms. Let $\{v_j^{(n)*}\}$ be the dual base of $\{v_j^{(n)}\}$, $< v_j^{(n)*}, v_k^{(n)} >= \delta_{jk}$. Then the following isomorphisms hold,

$$\begin{array}{rcl} (V_n)_{q^{\mp 2} z} & \simeq & (V_n)_z^{*a^{\pm 1}} \\ v_j^{(n)} & \mapsto & (-)^j q^{-j(n-j\mp 1)} \begin{bmatrix} n \\ j \end{bmatrix}^{-1} v_{n-j}^{(n)*}. \\ (-)^{n-j} q^{(n-j)(j\mp 1)} \begin{bmatrix} n \\ j \end{bmatrix} v_{n-j}^{(n)} & \leftarrow & v_j^{(n)*}. \end{array} \qquad (5)$$

2.5 Level one vertex opoerators

Let $V(\Lambda_i)$ be the irreducible highest weight $U_q'(\hat{s}l_2)$-module with highest weight $\Lambda_i (i=0,1)$, $\hat{V}(\Lambda_i)$ its weight completioin $\hat{V}(\Lambda_i)=\prod_{\nu\in P} V(\Lambda_i)_\nu$, $V(\Lambda_i)_\nu=\{v\in V(\Lambda_i)|q^h v=q^{<h,\nu>}v$ for $h\in P\}$ and $u^*_{\Lambda_i}$ the highest weight vector of the right module $V(\Lambda_i)^*$ such that $<u^*_{\Lambda_i},u_{\Lambda_i}>=1$. Let us denote $\Phi(z)$, $\Psi(z)$, $\Phi_V(z)$ and $\Psi_V(z)$ the $U_q'(\hat{s}l_2)$ intertwiners

$$
\begin{aligned}
&\Phi(z): V(\Lambda_i)\longrightarrow V(\Lambda_{i+1})\hat{\otimes}(V_1)_z,\\
&\Psi(z): V(\Lambda_i)\longrightarrow (V_1)_z\hat{\otimes}V(\Lambda_{i+1}),\\
&\Phi_V(z): V(\Lambda_i)\otimes(V_1)_z\longrightarrow \hat{V}(\Lambda_{i+1}),\\
&\Psi_V(z): (V_1)_z\otimes V(\Lambda_i)\longrightarrow \hat{V}(\Lambda_{i+1}),
\end{aligned}
$$

normalized as

$$
\begin{aligned}
&<u^*_{\Lambda_{i+1}},\Phi(z)u_{\Lambda_i}>=<u^*_{\Lambda_{i+1}},\Psi(z)u_{\Lambda_i}>=v^{(1)}_{1-i},\\
&<u^*_{\Lambda_{i+1}},\Phi_V(z)(u_{\Lambda_i}\otimes v^{(1)}_j)>=<u^*_{\Lambda_{i+1}},\Psi_V(z)(u_{\Lambda_i}\otimes v^{(1)}_j)>=\delta_{i,j}.
\end{aligned}
$$

Here and after, in general, we set $V(\Lambda_i)\hat{\otimes}(V_n)_z=(\prod_{\nu\in P}F[z,z^{-1}]\otimes V(\Lambda_i)_\nu)\otimes_{F[z,z^{-1}]}(V_n)_z$. In fact the images of $\Phi(z)$ and $\Psi(z)$ belong to smaller spaces[2].

The components of those operators are defined by

$$
\Phi_j(z)=<v^{(1)*}_j,\Phi(z)>,\quad \Psi_j(z)=<v^{(1)*}_j,\Psi(z)>.
$$

We shall also introduce the notations $\Phi^{V^{*a}}(z)$ and $\Psi^{V^{*a^{-1}}}(z)$ which are intertwiners

$$
\begin{aligned}
&\Phi^{V^{*a}}(z): V(\Lambda_i)\longrightarrow V(\Lambda_{i+1})\hat{\otimes}(V_1)^{*a}_z,\\
&\Psi^{V^{*a^{-1}}}(z): V(\Lambda_i)\longrightarrow (V_1)^{*a^{-1}}_z\hat{\otimes}V(\Lambda_{i+1}),
\end{aligned}
$$

defined by

$$
<v^{(1)}_j,\Phi^{V^{*a}}(z)u>=\Phi_V(z)(u\otimes v^{(1)}_j),\quad <v^{(1)}_j,\Psi^{V^{*a^{-1}}}(z)u>=\Psi_V(z)(v^{(1)}_j\otimes u).
$$

Those operators are related by

$$
\Phi^{V^{*a}}(z)=(-1)^{1-i}q^{-1+i}\Phi(q^{-2}z),\quad \Psi^{V^{*a^{-1}}}(z)=(-1)^{1-i}q^{1-i}\Psi(q^2z),
$$

under the isomorphism 5.

The commutation relations of those vertex operators are, on $V(\Lambda_i)$,

$$
\begin{aligned}
&-(\frac{z_1}{z_2})^i r(\frac{z_1}{z_2})\check{\bar{R}}(\frac{z_1}{z_2})\Phi(z_1)\Phi(z_2)=\Phi(z_2)\Phi(z_1),\\
&(\frac{z_1}{z_2})^{1-i} r(\frac{z_1}{z_2})\check{\bar{R}}(\frac{z_1}{z_2})\Psi(z_2)\Psi(z_1)=\Psi(z_1)\Psi(z_2),\\
&(\frac{z_1}{z_2})^{-1+i}\frac{\theta_{q^4}(\frac{qz_1}{z_2})}{\theta_{q^4}(\frac{qz_2}{z_1})}\Psi(z_1)\Phi(z_2)=\Phi(z_2)\Psi(z_1).
\end{aligned}
$$

We shall rewrite the first and second relations for the sake of later use as

$$q(\frac{z_1}{z_2})^i r(\frac{q^2 z_1}{z_2})\check{\bar{R}}(\frac{q^2 z_1}{z_2})\Phi(z_1)\Phi^{V^{*a}}(z_2) = \Phi^{V^{*a}}(z_2)\Phi(z_1),$$
$$-q^{-1}(\frac{z_1}{z_2})^{1-i} r(\frac{z_1}{q^2 z_2})\check{\bar{R}}(\frac{z_1}{q^2 z_2})\Psi^{V^{*a^{-1}}}(z_2)\Psi(z_1) = \Psi(z_1)\Psi^{V^{*a^{-1}}}(z_2).$$

Here $\check{\bar{R}}(z) = P\bar{R}(z)$, $P(u \otimes v) = v \otimes u$ and

$$\bar{R}(z)(v_j^{(1)} \otimes v_j^{(1)}) = v_j^{(1)} \otimes v_j^{(1)} \text{ for } j = 0, 1,$$
$$\bar{R}(z)(v_0^{(1)} \otimes v_1^{(1)}) = bv_0^{(1)} \otimes v_1^{(1)} + cv_1^{(1)} \otimes v_0^{(1)},$$
$$\bar{R}(z)(v_1^{(1)} \otimes v_0^{(1)}) = c'v_0^{(1)} \otimes v_1^{(1)} + bv_1^{(1)} \otimes v_0^{(1)},$$
$$b = \frac{1-z}{1-q^2 z}q, \quad c = \frac{1-q^2}{1-q^2 z}z, \quad c' = \frac{1-q^2}{1-q^2 z}, \quad r(z) = \frac{(z^{-1})_\infty (q^2 z)_\infty}{(z)_\infty (q^2 z^{-1})_\infty},$$

where $(z)_\infty = \prod_{j=0}^{\infty}(1 - zq^{4j})$.

Let us describe the inversion relations for vertex operators. Let us denote by P_F^1 the dual pairing map $(V_1)_z^{*a} \otimes (V_1)_z \longrightarrow F$ which is in fact a $U_q'(\hat{s}l_2)$ linear. Then we have

$$P_F^1 \Phi^{V^{*a}}(z)\Phi(z) = g^{-1}\mathrm{id}_{V(\Lambda_i)}, \tag{6}$$
$$Res_{z_1=z_2}\Psi(z_2)\Psi^{V^{*a^{-1}}}(z_1) = z_2 g w \otimes \mathrm{id}_{V(\Lambda_i)}, \tag{7}$$

where $g = \frac{(q^2)_\infty}{(q^4)_\infty}$ and

$$w = \sum_{j=0}^{1} v_j^{(1)*} \otimes v_j^{(1)} = v_0^{(1)} \otimes v_1^{(1)} - qv_1^{(1)} \otimes v_0^{(1)}$$

through the isomorphism 5. Note that 6 and 7 are equivalent, respectively, to

$$\Phi_V(z)\Phi(z) = g^{-1}\mathrm{id}_{V(\Lambda_i)}.$$
$$Res_{z_1=q^2 z_2}\Psi(z_2)\Psi(z_1) = (-1)^{1-i}q^{i+1}z_2 g w \otimes \mathrm{id}_{V(\Lambda_i)}. \tag{8}$$

2.6 Crystal

We shall review here the definitions and fundamental properties of crystals which we need in the subsequent sections. The details and generalities of this section can be found in [15, 13, 14].

Definition 1 *An affine crystal B is a set B with the weight decomposition $B = \sqcup_{\lambda \in P} B_\lambda$ and with the maps*

$$\tilde{e}_i, \tilde{f}_i : B \sqcup \{0\} \longrightarrow B \sqcup \{0\}$$

satisfying the following axioms:

(1) *$\tilde{e}_i B_\lambda \subset B_{\alpha_i+\lambda} \sqcup \{0\}$, $\quad \tilde{f}_i B_\lambda \subset B_{-\alpha_i+\lambda} \sqcup \{0\}$ for non-empty B_λ,*

(2) *$\tilde{e}_i 0 = \tilde{f}_i 0 = 0$,*

(3) *for any b and i, there exists n such that $\tilde{e}_i^n b = \tilde{f}_i^n b = 0$,*

(4) *for $b_1, b_2 \in B$, $b_2 = \tilde{f}_i b_1$ if and only if $b_1 = \tilde{e}_i b_2$,*

(5) *if we set*

$$\varphi_i(b) = max\{n|\tilde{f}_i^n b \in B\}, \quad \varepsilon_i(b) = max\{n|\tilde{e}_i^n b \in B\},$$

then $\varphi_i(b) - \varepsilon_i(b) =< h_i, \lambda >$ for $b \in B_\lambda$ and i.

Let us set $P_{cl} = P/\mathbf{Z}\delta$ and cl the projection $P \longrightarrow P_{cl}$. Then a classical crystal is defined using P_{cl} in stead of P in the definition of an affine crystal. In this paper crystal means affine or classical crystal. A slightly general definition of the concept of crystal is introduced in [13, 14].

A crystal has the structure of colored oriented graph by

$$b_1 \xrightarrow{i} b_2 \text{ if and only if } b_2 = \tilde{f}_i b_1.$$

A morphism $\psi : B^1 \longrightarrow B^2$ of the crystals is a map $B^1 \sqcup \{0\} \longrightarrow B^2 \sqcup \{0\}$ which commutes with the actions of $\tilde{e}_i$ and $\tilde{f}_i$ and satisfies $\psi(0) = 0$. A morphism of crystals is called isomorphism (injective) if the accociated map is bijective (injective). A crystal B_1 is called a subcrystal of B^2 if there is an injective morphism of crystals $B^1 \longrightarrow B^2$.

For a crystal B and a subset $I \subset \{0, 1\}$, the I-crystal B is the set B with the same weight decomposition and with the maps $\tilde{e}_j$, $\tilde{f}_j$ $(j \in I)$ which is a part of the maps of the original crystal B.

For two crystals B_1, B_2 we can define the tensor product in the following manner.

Definition 2 **(1)** *As a set $B^1 \otimes B^2 = \sqcup_{\lambda \in P}(B^1 \otimes B^2)_\lambda$, $(B^1 \otimes B^2)_\lambda = \sqcup_{\mu+\nu=\lambda} B^1_\mu \times B^2_\nu$. We denote (b_1, b_2) by $b_1 \otimes b_2$.*

(2) *The actions of $\tilde{e}_i$ and $\tilde{f}_i$ is defined as*

$$\tilde{f}_i(b_1 \otimes b_2) = \begin{cases} \tilde{f}_i b_1 \otimes b_2 & \varphi_i(b_1) > \varepsilon_i(b_2) \\ b_1 \otimes \tilde{f}_i b_2 & \varphi_i(b_1) \le \varepsilon_i(b_2) \end{cases}$$

$$\tilde{e}_i(b_1 \otimes b_2) = \begin{cases} \tilde{e}_i b_1 \otimes b_2 & \varphi_i(b_1) \ge \varepsilon_i(b_2) \\ b_1 \otimes \tilde{e}_i b_2 & \varphi_i(b_1) < \varepsilon_i(b_2) \end{cases}$$

Among the crystals we need, in this paper, three kinds of crystals. The first one is the crystal B_s associated with the crystal base of the representation $(V_s)_1$. More explicitly B_s can be described as

Definition 3 **(1)** *$B_s = \{\boxed{j} | 0 \le j \le s\}$ as a set.*

(2) $\tilde{f}_1\boxed{j} = \boxed{j\text{+}1}$ $\quad (0 \le j \le s-1), \quad \tilde{f}_0\boxed{j} = \boxed{j\text{-}1} \quad (1 \le j \le s), \quad \tilde{f}_i\boxed{j} = 0$ *otherwise.*

In the following we often use the notations $B_1 = \{\boxed{+}, \boxed{-}\}$ by the correspondence $\boxed{+} \leftrightarrow \boxed{0}$, $\boxed{-} \leftrightarrow \boxed{1}$ and identify $\pm$ with ± 1.

The second one is the affine crystal $Aff(B_s)$ which is called the affinization of B_s. $Aff(B_s)$ is defined as

$$Aff(B_s) = \sqcup_{\lambda \in P} Aff(B_s)_\lambda, \quad Aff(B_s)_\lambda = (B_s)_{cl(\lambda)}.$$

The actions of $\tilde{e}_i, \tilde{f}_i$ are specified by the commutative diagrams

$$\begin{array}{ccc} Aff(B_s)_\lambda & \xrightarrow{\tilde{e}_i} & Aff(B_s)_{\lambda+\alpha_i} \sqcup \{0\} \\ \| & & \| \\ (B_s)_{cl(\lambda)} & \xrightarrow{\tilde{e}_i} & (B_s)_{cl(\lambda+\alpha_i)} \sqcup \{0\} \end{array} \quad \text{and} \quad \begin{array}{ccc} Aff(B_s)_\lambda & \xrightarrow{\tilde{f}_i} & Aff(B_s)_{\lambda-\alpha_i} \sqcup \{0\} \\ \| & & \| \\ (B_s)_{cl(\lambda)} & \xrightarrow{\tilde{f}_i} & (B_s)_{cl(\lambda-\alpha_i)} \sqcup \{0\}. \end{array} \tag{9}$$

For example the graph of $Aff(B_1)$ is

$$\cdots \xrightarrow{1} \circ \xrightarrow{0} \circ \xrightarrow{1} \circ \xrightarrow{0} \circ \xrightarrow{1} \circ \xrightarrow{0} \cdots.$$

Third one is the crystal $B(\Lambda_i)$ associated with the crystal base of the representation $V(\Lambda_i)$. By now it is well known that $B(\Lambda_i)$ is described in terms of the set of paths[15, 9]. Let us define the space of paths $\mathcal{P}(\Lambda_i)$ as

$$\mathcal{P}(\Lambda_i) = \{(p(j))_{j=1}^{\infty} | p(j) \in B_1, p(k) = (-1)^{i+k} \text{ for } k >> 0\}$$

$\mathcal{P}(\Lambda_i)$ has the structure of an affine crystal by[9]

Theorem 1 **(1)** *There is an isomorphism of classical crystals,*

$$B(\Lambda_i) \simeq B(\Lambda_{1-i}) \otimes B_1. \tag{10}$$

(2) *The isomorphism 10 induces the bijective map $B(\Lambda_i) \simeq \mathcal{P}(\Lambda_i)$.*

The weight of a path through the above bijection can explicitly be written in terms of the energy function [15].

For an affine crystal B we define the dual crystal B^* of B as

Definition 4 **(1)** $B^* = \{b^\vee | b \in B\} = \sqcup_{\lambda \in P} B_{-\lambda}, \quad B_{-\lambda} = \{b^\vee | b \in B_\lambda\}$,

(2) $\tilde{e}_i b^\vee = (\tilde{f}_i b)^\vee, \quad \tilde{f}_i b^\vee = (\tilde{e}_i b)^\vee, \quad 0^\vee = 0.$

The map $(b_1 \otimes b_2)^\vee \mapsto b_2^\vee \otimes b_1^\vee$ gives the isomorphism

$$(B_1 \otimes B_2)^* \simeq B_2^* \otimes B_1^*.$$

Then we have the description of $B(\Lambda_i)^*$ in terms of paths,

$$B(\Lambda_i)^* = \{(p(j))_{j=-\infty}^{0} | p(j) \in B_1, p(k) = (-1)^{i+k} \text{ for } k << 0\},$$
$$B_1 \otimes B(\Lambda_i)^* \simeq B(\Lambda_{i+1})^*, \quad b \otimes (p(j))_{j=-\infty}^{0} \mapsto (p'(j))_{j=-\infty}^{0},$$

where $p'(0) = b$, $p'(j) = p(j+1)(j \le -1)$.

2.7 The morphism of crystals induced from the Dynkin diagram automorphism

Let ι be the isomorphism of the $\mathbf{Z}$ module P_{cl} defined by $\iota(\Lambda_i) = \Lambda_{1-i} (i = 0, 1)$. For a classical crystal B, we define the classical crystal $\iota^* B$ by

$$\iota^* B = \sqcup_{\lambda \in P_{cl}} (\iota^* B)_\lambda, \quad (\iota^* B)_\lambda = \{\iota^*(b) | b \in B_{\iota(\lambda)}\}, \quad , \iota(0) = 0, \tag{11}$$

$$\tilde{f}_i \iota^*(b) = \iota^*(\tilde{f}_{1-i} b), \quad \tilde{e}_i \iota^*(b) = \iota^*(\tilde{e}_{1-i} b). \tag{12}$$

It is easy to prove that 11, 12 actually defines a classical crystal. For this crystal the following properties hold.

Proposition 1 **(1)** *$\iota^* B(\Lambda_i) \simeq B(\Lambda_{1-i})$.*

(2) *$\iota^* B_s \simeq B_s$ by $\boxed{j} \mapsto \boxed{s\text{-}j}$.*

(3) *For crystals B^1, B^2, $B^1 \simeq B^2$ if and only if $\iota^* B^1 \simeq \iota^* B^2$.*

(4) *For crystals B^1, B^2, $\iota^*(B^1 \otimes B^2) \simeq \iota^* B^1 \otimes \iota^* B^2$, by $\iota^*(b_1 \otimes b_2) \mapsto \iota^*(b_1) \otimes \iota^*(b_2)$.*

The properties (2)-(4) can directly be checked using definitions. Let us prove (1). From 10, (2) and (4) above, we have

$$\iota^* B(\Lambda_i) \simeq \iota^* B(\Lambda_{1-i}) \otimes B_1, \quad \iota^*(u_{\Lambda_i}) \mapsto \iota^*(u_{\Lambda_{1-i}}) \otimes \boxed{(-1)^i}.$$

Using the fact that any element of $\iota^* B(\Lambda_i)$ can be written as $\tilde{f}_{j_1} \cdots \tilde{f}_{j_n} \iota^*(u_{\Lambda_i})$ for some n and $(j_1, \cdots, j_n) \in \{0, 1\}^n$, we have $\iota^* B(\Lambda_i) \simeq \mathcal{P}(\Lambda_{1-i})$ as a classical crystal. □

3 Isomorphisms of crystals

The structure of the space of the eigenvectors of the XXZ hamiltonian is, in the low temprature limit, described by the decomposition of the crystals of $B(\Lambda_i) \otimes B(\Lambda_j)^*$ [2]. In this section we shall prove a new type of isomorphisms of crystals which generalize Theorem 1 (1) and give a predicted form of the structure of the space of eigenvectors of our transfer matrix in the low temperature limit.

Our task is to decompose the crystals of the form

$$B(\Lambda_i) \otimes B_{s_1} \otimes \cdots \otimes B_{s_k} \otimes B(\Lambda_j)^*.$$

The main results in this section are

Theorem 2 *There is an isomorphism of classical crystals,*

$$B(\Lambda_i) \otimes B_s \simeq B_{s-1} \otimes B(\Lambda_{1-i}),$$

for $s = 1, 2, 3, \cdots$.

Corollary 1 *For $j = 0, 1$, we have the isomorphism of classical crystals,*

$$\coprod_{i=0,1} B(\Lambda_i) \otimes B_{s_1} \otimes \cdots \otimes B_{s_k} \otimes B(\Lambda_j)^* \simeq$$

$$B_{s_1-1} \otimes \cdots \otimes B_{s_k-1} \otimes \coprod_{n=0}^{\infty} \mathit{Aff}(B_1)^{\otimes n}/S_n.$$

Here the action of the symmetric group S_n on $\mathrm{Aff}(B_1)^{\otimes n}$ is not the usual one but that defined in [2].

The isomorphisms of Theorem 2 includes 10 as a special case $s = 1$. But the proof of Theorem 2 uses the isomorphism 10.

It is sufficient to prove the theorem for $i = 0$. Since $i = 1$ case is obtained by applying the map ι in subsection 2.7.

Let us define the map

$$\psi : B_{s-1} \otimes B(\Lambda_0) \longrightarrow B(\Lambda_1) \otimes B_s,$$

first and after that prove that it is well defined and commutes with the actions of $\tilde{e}_i$ and $\tilde{f}_j$. In order to define the map ψ we need to describe some isomorphisms.

Lemma 1 *There is an isomorphism of $\{0, 1\}$ crystals,*

$$\psi_1 : B_s \otimes B_1 \simeq B_1 \otimes B_s.$$

The isomorphism is given explicitly by

$$\begin{aligned}
&\boxed{j{+}1} \otimes \boxed{+} \longrightarrow \boxed{-} \otimes \boxed{j} \quad \textit{for } 0 \le j \le s-1,\\
&\boxed{0} \otimes \boxed{+} \longrightarrow \boxed{+} \otimes \boxed{0},\\
&\boxed{j} \otimes \boxed{-} \longrightarrow \boxed{+} \otimes \boxed{j{+}1} \quad \textit{for } 0 \le j \le s-1,\\
&\boxed{s} \otimes \boxed{-} \longrightarrow \boxed{-} \otimes \boxed{s}.
\end{aligned}$$

Using the map ψ_1 let us define the isomorphism

$$\psi_n : B_s \otimes B_1^{\otimes n} \simeq B_1^{\otimes n} \otimes B_s$$

by

$$\psi_n = (1_{n-1} \otimes \psi_1) \cdots (1_1 \otimes \psi_1 \otimes 1_{n-2})(\psi_1 \otimes 1_{n-1}),$$

where 1_j is the identity map of $B_1^{\otimes j}$.

Now let us define the map ψ in the following manner. Take any $\boxed{\mathrm{j}}_s \otimes b \in B_{s-1} \otimes B(\Lambda_0)$. For b there is $n \in \mathbf{Z}_{\ge 1}$ which satisfies

$$b = (b_k)_{k=1}^{\infty}, \quad b_k = (-1)^k \text{ for } k \ge 2n. \tag{13}$$

Taking any such n and set

$$\psi(\boxed{\mathrm{j}}_{s-1} \otimes b) = b_{\Lambda_0} \otimes \psi_{2n-1}(\boxed{\mathrm{j}}_s \otimes b_{2n-1} \otimes \cdots \otimes b_1)$$

throught the isomorphism

$$B(\Lambda_0) \otimes B_1^{\otimes 2n-1} \otimes B_s \simeq B(\Lambda_1) \otimes B_s,$$

where b_{Λ_0} is the highest weight element of $B(\Lambda_0)$ and the subscript of $\boxed{\mathrm{j}}$ specifies to which crystal the element belongs, $\boxed{\mathrm{j}}_s \in B_s$. The well definedness of ψ follows from

Lemma 2 *The definition of ψ does not depend on the choice of n which satisfies the condition 13.*

Proof
It is sufficient to prove

$$\boxed{+} \otimes \boxed{-} \otimes \psi_n(\boxed{\mathrm{j}}_s \otimes b') = \psi_{n+2}(\boxed{\mathrm{j}}_s \otimes \boxed{-} \otimes \boxed{+} \otimes b'),$$

for $0 \le j \le s-1$, $n \in \mathbf{Z}_{\ge 1}$ and any $b' \in B_1^{\otimes n}$. These equations follow from Lemma 1. □

Lemma 3 *The map ψ commutes with the actions of $\tilde{e}_1$ and $\tilde{f}_1$.*

Proof
Let B be the connected component, as a $\{1\}$-crystal, of $B_{s-1} \otimes B_1$ which contains $\boxed{0}_{s-1} \otimes \boxed{+}$. Then

$$B = \{\boxed{\mathrm{j}}_{s-1} \otimes \boxed{+} \mid 0 \le j \le s-1\} \sqcup \{\boxed{\mathrm{s\text{-}1}}_{s-1} \otimes \boxed{-}\}$$

and B is isomorphic to B_s as a $\{1\}$-crystal by the map

$$\begin{array}{ccl} B & \longrightarrow & B_s \\ \boxed{\mathrm{j}}_{s-1} \otimes \boxed{+} & \mapsto & \boxed{\mathrm{j}}_s \text{ for } 0 \le j \le s-1 \\ \boxed{\mathrm{s\text{-}1}}_{s-1} \otimes \boxed{-} & \mapsto & \boxed{\mathrm{s}}_{\frac{s}{2}}. \end{array}$$

Let $\boxed{\mathrm{j}}_s \otimes b \in B_{s-1} \otimes B(\Lambda_0)$ and n as above. Now we shall describe ψ as a composition of several crystal morphisms from the connected component of $\boxed{\mathrm{j}}_s \otimes b$ as a $\{1\}$-crystal. First of all

$$\begin{array}{rcl} B_s \otimes B(\Lambda_0) & \simeq & B_{s-1} \otimes B(\Lambda_0) \otimes B_1^{\otimes 2n} \\ \boxed{\mathrm{j}}_{s-1} \otimes b & \mapsto & \boxed{\mathrm{j}}_{s-1} \otimes b_{\Lambda_0} \otimes \boxed{+} \otimes b_{2n-1} \otimes \cdots b_1 =: \tilde{b} \end{array}$$

is an isomorphism of classical crystals. The crystal $B \otimes B_1^{\otimes 2n-1}$ is a sub $\{1\}$-crystal of $B_{s-1} \otimes B(\Lambda_0) \otimes B_1^{\otimes 2n}$, by the map

$$\boxed{\mathrm{j}}_{s-1} \otimes \boxed{\epsilon} \longrightarrow \boxed{\mathrm{j}}_{s-1} \otimes b_{\Lambda_0} \otimes \boxed{\epsilon} \otimes b_{2n-1} \otimes \cdots \otimes b_1.$$

The element $\tilde{b}$ is in this subcrystal. Next

$$B \otimes B_1^{\otimes 2n-1} \simeq B_s \otimes B_1^{\otimes 2n-1}$$

as a $\{1\}$-crystal as we already discussed. We have the isomorphism of $\{0,1\}$-crystal

$$\psi_{2n-1} : B_s \otimes B_1^{\otimes 2n-1} \simeq B_1^{\otimes 2n-1} \otimes B_s.$$

Finally we have the injective $\{1\}$-crystal morphism

$$\begin{array}{ccc} B_1^{\otimes 2n-1} \otimes B_s & \longrightarrow & B(\Lambda_0) \otimes B_1^{\otimes 2n-1} \otimes B_s \\ b' & \mapsto & b_{\Lambda_0} \otimes b'. \end{array}$$

It is easy to check that the map ψ is the composition of the above maps. Since we can take sufficiently large n such that the condition 13 holds for For $\boxed{\text{j-1}}_{s-1} \otimes b$, $\tilde{f}_1(\boxed{\text{j-1}}_{s-1} \otimes b)$ and $\tilde{e}_1(\boxed{\text{j-1}}_{s-1} \otimes b)$, ψ is a $\{1\}$-crystal morphism. $\Box$

Lemma 4 *The map ψ commutes with the action of $\tilde{e}_0$ and $\tilde{f}_0$.*

Proof
Let us define a map $\tilde{\psi}$ in the following manner. For $\boxed{\text{j}}_{s-1} \otimes b \in B_{s-1} \otimes B(\Lambda_0)$, take $n \in \mathbf{Z}_{\geq 0}$ such that

$$b = (b_k)_{k=1}^{\infty}, \quad b_k = (-1)^k \text{ for } k \geq 2n+1.$$

Then

$$\tilde{\psi}(\boxed{\text{j}}_{s-1} \otimes b) = b_{\Lambda_1} \otimes \psi_{2n}(\boxed{\text{j+1}}_{s} \otimes b_{2n} \otimes \cdots \otimes b_1)$$

through the isomorphism

$$B(\Lambda_1) \otimes B_1^{2n} \otimes B_s \simeq B(\Lambda_1) \otimes B_s.$$

In a similar manner to the ψ case, we can check that the definition of $\tilde{\psi}$ is independent of the choice of n.

Sublemma 1 $\psi = \tilde{\psi}$.

Proof
We use the above notations. Take n in such a way that satisfies the condition 13. Then

$$\psi(\boxed{\text{j}}_{s-1} \otimes b) = \tilde{\psi}(\boxed{\text{j}}_{s-1} \otimes b)$$

is equivalent to

$$\boxed{\text{-}} \otimes \psi_{2n-1}(\boxed{\text{j}}_{s} \otimes b_{2n-1} \otimes \cdots \otimes b_1) = \psi_{2n}(\boxed{\text{j+1}}_{s} \otimes \boxed{+} \otimes b_{2n-1} \otimes \cdots \otimes b_1)$$

for $0 \leq j \leq s-1$. This follows from Lemma 1. $\Box$

Now the commutativity of $\tilde{\psi}$ and the action of $\tilde{f}_0$ and $\tilde{e}_0$ is similarly proved as before. Namely let us set

$$B' = \{\boxed{\text{j}}_{s-1} \otimes \boxed{\text{-}} \mid 0 \leq j \leq s-1\} \sqcup \{\boxed{0} \otimes \boxed{+}\}.$$

Then this constitutes, as a $\{0\}$-crystal, a connected component of $B_{s-1} \otimes B_1$ isomorphic to B_s. The map is given by

$$\begin{array}{rcl} B' & \longrightarrow & B_s \\ \boxed{\mathrm{j}}_{s-1} \otimes \boxed{-} & \mapsto & \boxed{\mathrm{j+1}}_s \text{ for } o \leq j \leq s-1 \\ \boxed{0}_{\frac{s}{2}} \otimes \boxed{+} & \mapsto & \boxed{0}_{\frac{s}{2}}. \end{array}$$

Using this description it is easy to show that the $\tilde{\psi}$ is described as a composition of $\{0\}$ crystal morphisms from any $\{0\}$-crystal connected component as before. Hence the lemma is proved. □

Lemma 5 *ψ is a bijection.*

Proof
We shall prove the injectivity first. Suppose that

$$\psi(\boxed{\mathrm{j}}_{s-1} \otimes b) = \psi(\boxed{\mathrm{j'}}_{\frac{s-1}{2}} \otimes b').$$

By the definition of ψ this is equivalent to

$$\psi_{2n-1}(\boxed{\mathrm{j}}_s \otimes b_{2n-1} \otimes \cdots \otimes b_1) = \psi_{2n-1}(\boxed{\mathrm{j'}}_s \otimes b'_{2n-1} \otimes \cdots \otimes b'_1),$$

for sufficiently large n. Since ψ_{2n-1} is bijective, we have

$$j = j', \quad b_k = b'_k \text{ for } 1 \leq k \leq 2n-1$$

which means $b = b'$. The surjectivity easily follows from Lemma 1:

$$\begin{array}{l} \psi_1^{-1}(\boxed{+} \otimes \boxed{\mathrm{j+1}}_s) = \boxed{\mathrm{j}}_s \otimes \boxed{-} \text{ for } 0 \leq j \leq s-1 \\ \psi_1^{-1}(\boxed{-} \otimes \boxed{\mathrm{j}}_s) = \boxed{\mathrm{j+1}}_s \otimes \boxed{+} \text{ for } 0 \leq j \leq s-1, \\ \psi_1^{-1}(\boxed{-} \otimes \boxed{\mathrm{s}}_{\frac{s}{2}}) = \boxed{\mathrm{s}}_{\frac{s}{2}} \otimes \boxed{-}. \end{array}$$

□
This lemma complets the proof of theorem 2.

4 Existence of new type of vertex operators

In this section we shall prove the existence of new types of q-vertex operators, one type of which is conjectured to induce the crystal isomorphisms in section 3. For non-zero complex numbers $z_1, \cdots, z_k$ and $(i,j) \in \{0,1\}^2$ let us define the $F[z_1^{\pm 1}, \cdots, z_k^{\pm 1}]$ module by

$$H_{z_1 \cdots z_k}^{n_1 \cdots n_k}(i,j) = \{v \in (V_{n_1})_{z_1} \otimes \cdots \otimes (V_{n_k})_{z_k} | wt(v) = \Lambda_i - \Lambda_j, e_l^{<h_l, \Lambda_j>+1} v = 0 \text{ for } l = 0,1\}.$$

Our aim here is to prove

Theorem 3 **(1)** *$H^{n,m}_{z_2,q^{-3}z_1}(i,i+1)$, $H^{n,n}_{q^2z_1,z_2}(i,i)$ and $H^{n+1,1,n}_{z_1,z_2,q^{-3}z_1}(i,i+1)$ are free $F[z_1^{\pm 1},z_2^{\pm 1}]$ modules and their ranks are given by*

$$\begin{aligned}
&rank H^{n,m}_{z_2,q^{-3}z_1}(i,i+1)=\delta_{|n-m|,1}\delta_{z_1,z_2},\\
&rank H^{n,n}_{q^2z_1,z_2}(i,i)=\delta_{z_1,z_2},\\
&rank H^{n+1,1,n}_{z_1,z_2,q^{-3}z_1}(i,i+1)=1.
\end{aligned}$$

(2) *There are isomorphisms of $F[z_1^{\pm 1},z_2^{\pm 1}]$ modules*

$$\begin{aligned}
&Hom_{F'\otimes U'}((V_m)_{z_1}\otimes V(\Lambda_i),V(\Lambda_{i+1})\hat{\otimes}(V_n)_{z_2})\simeq H^{n,m}_{z_2,q^{-3}z_1}(i,i+1),\\
&Hom_{F'\otimes U'}((V_n)_{z_1}\otimes V(\Lambda_i),(V_n)_{z_2}\hat{\otimes}V(\Lambda_i))\simeq H^{n,n}_{q^2z_1,z_2}(i,i),\\
&Hom_{F'\otimes U'}((V_n)_{z_1}\otimes V(\Lambda_i),V(\Lambda_{i+1})\hat{\otimes}(V_{n+1})_{z_1}\otimes(V_1)_{z_2})\simeq H^{n+1,1,n}_{z_1,z_2,q^{-3}z_1}(i,i+1),
\end{aligned}$$

where $F'=F[z_1^{\pm 1}]$.

Corollary 2

$$\begin{aligned}
&Hom_{U'}(V(\Lambda_i),(V_n)_{q^2z}\hat{\otimes}V(\Lambda_{i+1})\hat{\otimes}(V_{n+1})_z)\\
&\simeq Hom_{F[z^{\pm 1}]\otimes U'}(V(\Lambda_i)\otimes(V_{n+1})_z,(V_n)_z\hat{\otimes}V(\Lambda_{i+1}))\simeq F[z^{\pm 1}].
\end{aligned}$$

Let us prove (1). Other cases are similarly proved. Note that

$$\begin{aligned}
&\mathrm{Hom}_{U'}((V_m)_{z_1}\otimes V(\Lambda_i),V(\Lambda_{i+1})\hat{\otimes}(V_n)_{z_2})\\
&\simeq \mathrm{Hom}_{U'}(V(\Lambda_i),(V_m)_{q^2z_1}\hat{\otimes}V(\Lambda_{i+1})\hat{\otimes}(V_n)_{z_2}).
\end{aligned}$$

Let $U'(b_+)$ be the subalgebra of U' generated by e_i, t_i $(i=0,1)$. Then we have

$$\begin{aligned}
&\mathrm{Hom}_{U'}(V(\Lambda_i),(V_m)_{q^2z_1}\hat{\otimes}V(\Lambda_{i+1})\hat{\otimes}(V_n)_{z_2})\\
&\simeq \mathrm{Hom}_{U'(b_+)}(\mathbf{Q}(q)u_{\Lambda_i},(V_m)_{q^2z_1}\hat{\otimes}V(\Lambda_{i+1})\hat{\otimes}(V_n)_{z_2})\\
&\simeq \mathrm{Hom}_{U'(b_+)}(V(\Lambda_{i+1})^{*a}\otimes(V_m)_{z_1}\otimes\mathbf{Q}(q)u_{\Lambda_i},(V_n)_{z_2}).
\end{aligned}$$

Here we used the following lemma which can be proved in a similar way to that in [3].

Lemma 6 *Take any i and fix it. Let $u\in(V_n)_z\hat{\otimes}V(\Lambda_i)\hat{\otimes}(V_m)_z$ be a weight vector of t_i. If u satisfies $e_i^l u=0$ for some l, then $f_i^N u=0$ for some N.*

Lemma 7 *There is an isomorphism of $U'(b_+)$-modules,*

$$(V_n)_z\otimes Fu_{\Lambda_i}\simeq Fu_{\Lambda_i}\otimes(V_n)_{q^{-1}z}$$

given by the map

$$v_j^{(n)}\otimes u_{\Lambda_i}\longrightarrow q^{-ji}u_{\Lambda_i}\otimes v_j^{(n)}.$$

This lemma implies

$$
\begin{aligned}
&\mathrm{Hom}_{U'(b_+)}(V(\Lambda_{i+1})^{*a} \otimes (V_m)_{z_1} \otimes \mathbf{Q}(q)u_{\Lambda_i}, (V_n)_{z_2}) \\
&\simeq \mathrm{Hom}_{U'(b_+)}(V(\Lambda_{i+1})^{*a} \otimes \mathbf{Q}(q)u_{\Lambda_i}, (V_n)_{z_2} \otimes (V_m)_{q^{-3}z_1}) \\
&\simeq H^{n,m}_{z_2,q^{-3}z_1}(i,i+1). \qquad (14)
\end{aligned}
$$

Let us write explicitly the conditions satisfied by the vector v of $H^{n,m}_{z_2,q^{-3}z_1}(i,i+1)$ according to $i=0,1$;

$$wt(v) = \Lambda_0 - \Lambda_1, \quad e_1^2 v = e_0 v = 0, \text{ if } i=0 \qquad (15)$$

$$wt(v) = \Lambda_1 - \Lambda_0, \quad e_1 v = e_0^2 v = 0, \text{ if } i=1. \qquad (16)$$

Let us determine the vectors which satisfies the condition 15 and 16. Note first that the condition 15 or 16 implies $|n-m|=1$. In fact the vector satisfying 15 or 16 must lie in the two dimensional irreducible representation of $U'_q((\hat{s}l_2)_i)$.

Let w_j be the highest weight vectors of $V_n \otimes V_m$ with the weight $(n+m-2j)\Lambda_1$ as a $U'_q((\hat{s}l_2)_1)$-module. They are explicitly given by

$$w_j = \sum_{k=0}^{j} c_k^{(j)}(n) v_k^{(n)} \otimes v_{j-k}^{(m)}, \qquad (17)$$

$$c_k^{(j)}(n) = (-1)^k q^{k(n+1-k)} \begin{bmatrix} j \\ k \end{bmatrix}, \quad c_0^{(0)}(n) = 1. \qquad (18)$$

(1.1) $i=0$ and $n=m+1$ case
The vector satisfying the condition 15 is proportional to $f_1 w_m$. Let us calculate $e_0 f_1 w_m$ in the tensor product $(V_{m+1})_{z_2} \otimes (V_m)_{q^{-3}z_1}$. The result is

$$f_1 w_m = \sum_{k=1}^{m+1} c_{k-1}^{(m)}(m+1) q^{m-2k+3}(q^{-1}[m+2-k] - [m+1-k]) v_k^{(m+1)} \otimes v_{m+1-k}^{(m)}$$

$$e_0 f_1 w_m = (z_2 - z_1) \sum_{k=2}^{m+1} c_{k-2}^{(m)}(m+1) q^{-k+2}[m+2-k] v_k^{(m+1)} \otimes v_{m-k+2}^{(m)}.$$

Hence $e_0 v = 0$ is equivalent to $z_1 = z_2$.
(1.2) $i=0$ and $m=n+1$ case
The vector satisfying the condition 15 is proportional to $f_1 w_n$. We have

$$f_1 w_n = \sum_{k=0}^{n} c_k^{(n)}(n)(-q^{-1}[k] + [k+1]) v_k^{(n)} \otimes v_{n+1-k}^{(n+1)},$$

$$e_0 f_1 w_n = (z_1 - z_2) \sum_{k=1}^{n} c_k^{(n)}(n) q^{-n+3(k-1)}[k] v_k^{(n)} \otimes v_{n+2-k}^{(n+1)}.$$

Hence $e_0 f_1 w_n = 0$ is equivalent to $z_1 = z_2$.
(1.3) $i=1$ and $n=m+1$ case

The vector satisfying the condition 16 is proportional to w_m. Then

$$e_0^2 w_m = (z_1 - z_2) \sum_{k=2}^{m+1} c_{k-2}^{(m)} (m+1)(z_1[m+1-k] - z_2[m+3-k]) v_k^{(m+1)} \otimes v_{m+2-k}^{(m)}.$$

Therefore $e_0^2 w_m = 0$ if and only if $z_1 = z_2$.
(1.4) $i = 1$ and $m = n+1$ case
The vector satisfying the condition 16 is proportional to w_n. Then we have

$$e_0^2 w_n = q^{-6}(z_1 - z_2) \sum_{k=1}^{n} c_k^{(n)} (n) q^{-2n+4k} (z_1[k+1] - z_2[k-1]) v_k^{(n)} \otimes v_{n+2-k}^{(n+1)}.$$

Consequently $e_0^2 w_n = 0$ iff $z_1 = z_2$. $\square$

By theorem 3 there are uniquely determined $U_q'(\hat{s}l_2)$ intertwinwers

$$\begin{aligned} {}_{V_n}\Phi^{V_{n+1}}(z) &: (V_n)_z \otimes V(\Lambda_i) \longrightarrow V(\Lambda_{i+1}) \hat{\otimes} (V_{n+1})_z, \\ {}^{V_n}\Phi^{V_{n+1}}(z) &: V(\Lambda_i) \longrightarrow (V_n)_{q^2 z} \otimes V(\Lambda_{i+1}) \hat{\otimes} (V_{n+1})_z, \\ {}^{V_n}\Phi_{V_{n+1}}(z) &: V(\Lambda_i) \otimes (V_{n+1})_z \longrightarrow (V_n)_z \hat{\otimes} V(\Lambda_{i+1}), \end{aligned}$$

under the normalizations

$$\begin{aligned} &< u_{\Lambda_{i+1}}^*, {}_{V_n}\Phi^{V_{n+1}}(z)(v_j^{(n)} \otimes u_{\Lambda_i}) >= v_{1-i+j}^{(n+1)}, \\ &< v_j^{(n)*} \otimes u_{\Lambda_{i+1}}^*, {}^{V_n}\Phi^{V_{n+1}}(z)(u_{\Lambda_i}) >= (-1)^j q^{j(n+1-j)} \left[\begin{array}{c} n \\ j \end{array} \right] v_{n+1-j-i}^{(n+1)}, \\ &< u_{\Lambda_{i+1}}^*, {}^{V_n}\Phi_{V_{n+1}}(z)(u_{\Lambda_i} \otimes v_j^{(n+1)}) >= q^{(2i-1)j-i} \frac{[j]^i [n+1-j]^{1-i}}{[n+1]^{1-i}} v_{j-i}^{(n)} \quad (i \le j \le n+i). \end{aligned}$$

In the following we also call those intertwiners simply vertex operators.

Remark 1 *I conjecture that the vertex opertator ${}_{V_n}\Phi^{V_{n+1}}(z)$ preserves the crystal lattice and induces the isomorphism of crystals in section 3. Some part of Miki's conjecture[18] is a special case of this conjecture.*

5 Fusion of representations

Let us briefly recall the fusion construction of representations and R-matrices in order to fix notations. Let M_i be the trivial representation in the tensor product $(V_1)_{q^{-2(i-1)}z} \otimes (V_1)_{q^{-2i}z}$ for $1 \le i \le n$. Explicitly M_i is written

$$M_i = F \cdot (v_0^{(1)} \otimes v_1^{(1)} - q v_1^{(1)} \otimes v_0^{(1)}).$$

Let us set $N_i = (V_1)_z \otimes \cdots \otimes M_i \otimes \cdots \otimes (V_1)_{q^{-2n}z}$ and

$$\begin{aligned} W_{n+1}(z) &= (V_1)_z \otimes \cdots \otimes (V_1)_{q^{-2n}z} / \sum_{i=1}^{n} N_i, \\ \tilde{W}_{n+1}(z) &= U_q'(\hat{s}l_2)(v_0^{(1)} \otimes \cdots \otimes v_0^{(1)}) \hookrightarrow (V_1)_{q^{-2n}z} \otimes \cdots \otimes (V_1)_z \end{aligned}$$

Then the following proposition is well known.

Proposition 2 $W_{n+1}(z) \simeq \tilde{W}_{n+1}(z) \simeq (V_{n+1})_{q^{-n}z}$.

In order to describe the isomorphism explicitly we shall introduce the following definitions.

Definition 5 **(1)** *$(\epsilon_1, \cdots, \epsilon_n)$ is of type j if and only if $\sharp\{k|\epsilon_k = 1\} = j$.*

(2) *For $(\epsilon_1, \cdots, \epsilon_n)$ let us define its inversion number by*

$$inv(\epsilon_1, \cdots, \epsilon_n) = \sum_{i:\epsilon_i=1} \sharp\{k|\epsilon_k = 0, k < i\}.$$

Then the isomorphism is given by

$$\begin{array}{ccc} W_{n+1}(z) & \longrightarrow & (V_{n+1})_{q^{-n}z} \\ v^{(1)}_{\epsilon_1} \otimes \cdots \otimes v^{(1)}_{\epsilon_n} & \mapsto & q^{\mathrm{inv}(\epsilon_1,\cdots,\epsilon_n)} v^{(n+1)}_j \end{array}$$

for $(\epsilon_1, \cdots, \epsilon_n)$ of type j.

Let us give the description of $\tilde{W}_{n+1}$ in terms of R-matrix for the later use. Let $\check{\check{R}}(\frac{z_1}{z_2})$ be the $U'_q(\hat{s}l_2)$ linear morphism $(V_1)_{z_1} \otimes (V_1)_{z_2} \longrightarrow (V_1)_{z_2} \otimes (V_1)_{z_1}$ such that $\check{\check{R}}(\frac{z_1}{z_2})(v_0^{(1)\otimes 2}) = v_0^{(1)\otimes 2}$. Consider the composition $\check{\check{R}}_{n+1}(z) = \check{\check{R}}(\frac{z_n}{z_{n+1}}) \cdots \check{\check{R}}(\frac{z_1}{z_{n+1}}) \cdots \check{\check{R}}(\frac{z_1}{z_2})$ at $z_j = q^{-2(j-1)}z (1 \le j \le n+1)$. Then it is well known (and easily proved) that

Proposition 3

$$Im \check{\check{R}}_{n+1}(z) = \tilde{W}_{n+1}(z), \quad Ker \check{\check{R}}_{n+1}(z) = \sum_{k=1}^{n} N_k,$$

5.1 Fusion of the R-matrix

Let $\check{\check{R}}_{n+1,1}(\frac{z}{w}) = \check{\check{R}}(\frac{q^n z}{w}) \check{\check{R}}(\frac{q^{n-2}z}{w}) \cdots \check{\check{R}}(\frac{q^{-n}z}{w})$ be the $U'_q(\hat{s}l_2)$ intertwiner $(V_1)_{q^n z} \otimes \cdots (V_1)_{q^{-n}z} \otimes (V_1)_w \longrightarrow (V_1)_w \otimes (V_1)_{q^n z} \otimes \cdots (V_1)_{q^{-n}z}$. Then

Proposition 4 *$\check{\check{R}}_{n+1,1}(\frac{z}{w})$ induces the $U'_q(\hat{s}l_2)$ linear map $W_{n+1}(q^n z) \otimes (V_1)_w \longrightarrow (V_1)_w \otimes W_{n+1}(q^n z)$ such that the following diagram is commutative:*

$$\begin{array}{ccc} (V_1)_{q^n z} \otimes \cdots (V_1)_{q^{-n}z} \otimes (V_1)_w & \stackrel{\check{\check{R}}_{n+1,1}(\frac{z}{w})}{\longrightarrow} & (V_1)_w \otimes (V_1)_{q^n z} \otimes \cdots (V_1)_{q^{-n}z} \\ \downarrow & & \downarrow \\ W_{n+1}(q^n z) \otimes (V_1)_w & \longrightarrow & (V_1)_w \otimes W_{n+1}(q^n z). \end{array}$$

Here the downarrows are the natural projections.

Proof: It is sufficient to prove

$$\check{\check{R}}_{n+1,1}(\frac{z}{w})(N_j \otimes (V_1)_w) \subset (V_1)_w \otimes \sum_{k=1}^{n} N_k.$$

By Proposition 3 this is equivalent to

$$\check{\bar{R}}_{n+1}(q^n z)\check{\bar{R}}_{n+1,1}(\frac{z}{w})(N_j \otimes (V_1)_w) = 0$$

which follows from the Yang-Baxter equation. $\square$

We use the same symbol for $\check{\bar{R}}_{n+1,1}(\frac{z}{w})$ for the induced map. Then, explicitly, $\check{\bar{R}}_{n+1,1}(z)$ is given by

$$\check{\bar{R}}_{n,1}(z)\left[\begin{array}{c} v_k^{(n)} \otimes v_1^{(1)} \\ v_{k+1}^{(n)} \otimes v_0^{(1)} \end{array}\right] = \frac{1}{1-q^{n+1}z}\left[\begin{array}{cc} -q^{k+1}z + q^{n-k} & (1-q^{2n-2k})z \\ 1-q^{2k+2} & -q^{n-k}z + q^{k+1} \end{array}\right]\left[\begin{array}{c} v_1^{(1)} \otimes v_k^{(n)} \\ v_0^{(1)} \otimes v_{k+1}^{(n)} \end{array}\right].$$

6 Fusion of q-vertex operators

In this section we shall give a construction of the vertex operator ${}_{V_n}\Phi^{V_{n+1}}(z)$ in terms of $\Phi(z)$ and $\Psi(z)$. The idea is to consider the composition

$$\begin{array}{rcl} V(\Lambda_i) & \longrightarrow & (V_1)_{q^{n+1}z} \otimes \cdots \otimes (V_1)_{q^{-n+3}z} \otimes V(\Lambda_{i+1}) \otimes \otimes (V_1)_{q^n z} \otimes \cdots \otimes (V_1)_{q^{-n}z} \\ & & \downarrow \\ & & (V_n)_{q^{-n+2}z} \otimes V(\Lambda_{i+1}) \otimes (V_{n+1})_{q^{-n}z}. \end{array}$$

For the sake of simplicity hereafter we omit writting the the symbol ^ of the extended tensor product. The vertical arrow is the $U_q'(\hat{s}l_2)$-linear projection defined in Proposition 2. Unfortunately the composition of vertex operators Φ and Ψ which gives the horizontal arrow is not well defined in general. So we must carefully proceed in the following manner. Let us define the operator $O(\mathbf{z}|\mathbf{u})$, $(\mathbf{z},\mathbf{u}) \in \mathbf{C}^{*n+1} \times \mathbf{C}^{*n}$, acting on $V(\Lambda_i)$ by

$$O(z_1,\cdots,z_{n+1}|u_n,\cdots,u_1) = \frac{1}{f}\Phi(z_1)\cdots\Phi(z_{n+1})\Psi(u_n)\cdots\Psi(u_1),$$

where $\mathbf{C}^* = \{z \in \mathbf{C} | z \neq 0\}$ and

$$f(z_1,\cdots,z_{n+1}|u_n,\cdots,u_1) = \prod_{j<k}\frac{(\frac{q^2 z_k}{z_j})_\infty}{(\frac{q^4 z_k}{z_j})_\infty}\prod_{j>k}\frac{(\frac{u_k}{u_j})_\infty}{(\frac{q^2 u_k}{u_j})_\infty}\prod_{j,k}\frac{(\frac{q u_j}{z_k})_\infty}{(\frac{u_j}{q z_k})_\infty}.$$

The operator $O(\mathbf{z}|\mathbf{u})$ satisfies, on $V(\Lambda_i)$, the symmetry relations

$$\begin{array}{l} (\frac{z_j}{z_{j+1}})^{-1+\overline{i+j}}\check{\bar{R}}(\frac{z_j}{z_{j+1}})O(\mathbf{z}|\mathbf{u}) = O(\sigma_j\mathbf{z}|\mathbf{u}), \\ (\frac{u_j}{u_{j+1}})^{\overline{i+j}}\check{\bar{R}}(\frac{u_j}{u_{j+1}})O(\mathbf{z}|\mathbf{u}) = O(\mathbf{z}|\sigma_j\mathbf{u}), \end{array}$$

where σ_j is the permutation exchanging z_j, z_{j+1} or u_j, u_{j+1} and $\overline{k} = 0, 1$ according to k is even or odd. Let

$$\begin{array}{l} Pr(z)_{jk} : (V_1)_{z_j} \otimes (V_1)_{z_k} \longrightarrow V_2, \\ Pr(u)_{jk} : (V_1)_{u_j} \otimes (V_1)_{u_k} \longrightarrow V_2, \\ Pr(z) : (V_1)_{z_1} \otimes \cdots \otimes (V_1)_{z_{n+1}} \longrightarrow V_{n+1}, \\ Pr(u) : (V_1)_{u_1} \otimes \cdots \otimes (V_1)_{u_n} \longrightarrow V_n, \end{array}$$

be the $U_q'((\hat{sl}_2)_1)$-linear projection normalized as

$$Pr(z)_{jk}(v_0^{(1)\otimes 2}) = v_0^{(2)},$$
$$Pr(z)(v_0^{(1)\otimes n+1}) = v_0^{(n+1)}$$

and similarly for $Pr(u)_{jk}$, $Pr(u)$. Since $Pr(z)$ and $Pr(u)$ is determined uniquely under these normalizations, we have, for $j<k$

$$Pr(z) = Pr(z)\check{\bar{R}}(\frac{z_j}{z_{k-1}})\cdots\check{\bar{R}}(\frac{z_j}{z_{j+1}}). \tag{19}$$

The $Pr(z)$ in the right hand side is the $U_q'((\hat{sl}_2)_1)$ linear projection

$$(V_1)_{z_1}\otimes\cdots\otimes(V_1)_{z_j}\otimes(V_1)_{z_k}\otimes\cdots\otimes(V_1)_{z_{n+1}} \longrightarrow V_{n+1}.$$

To simplify the notations we use the same symbol $Pr(z)$ althought the space acted by it is different from that of $Pr(z)$ in the left hand side. Note that there is an $U_q'((\hat{sl}_2)_1)$-linear projection such that

$$\begin{array}{ccc}
(V_1)_{z_1}\otimes\cdots\otimes(V_1)_{z_j}\otimes(V_1)_{z_{j+1}}\otimes\cdots\otimes(V_1)_{z_{n+1}} & \overset{Pr(z)_{jk}}{\longrightarrow} & (V_1)_{z_1}\otimes\cdots\otimes V_2\otimes\cdots\otimes(V_1)_{z_{n+1}} \\
\downarrow & & \downarrow Pr(z)^{jk} \\
V_{n+1} & = & V_{n+1}
\end{array}$$

is a commutative diagram.

Proposition 5 **(1)** *The operator $O(\mathbf{z}|\mathbf{u})$ has poles at most simple at $z_j = q^2 z_k$ $(j<k)$ and $u_j = q^2 u_k$ $(j<k)$.*

(2) *The operator $Pr(z)Pr(u)O(\mathbf{z}|\mathbf{u})$ is regular at $z_j = q^2 z_k$ $(j<k)$ and $u_j = q^2 u_k$ $(j<k)$.*

Proof:(1) The integral formula of $<\Lambda_{i+1}|O(\mathbf{z}|\mathbf{u})|\Lambda_i>$ (appendix?) implies that $O(\mathbf{z}|\mathbf{u})$ has poles at most at $z_j = q^2 z_k$ $(j<k)$, $u_j = q^2 u_k$ $(j<k)$ and $u_j = q z_k, q^3 z_k$ for any j,k. Because there is a possibility to occur a pinch of the integration pathes only in those cases. Moreover it is easy to prove that these poles are at most simple. Hence it is sufficient to prove that there are no poles at $u_j = q z_k, q^3 z_k$ for any j,k. But again this follows easily from the integral formula of $<\Lambda_{i+1}|O(\mathbf{z}|\mathbf{u})|\Lambda_i>$ by the following reason. Consider a component of $<\Lambda_{i+1}|O(\mathbf{z}|\mathbf{u})|\Lambda_i>$. Let us decompose each integral as

$$\int_{C_d}\frac{d\xi_d}{2\pi i} = \int_{C_0}\frac{d\xi_d}{2\pi i} + \sum_{j=1}^{d} Res_{\xi_d=u_j}, \quad \int_{C_a}\frac{dw_a}{2\pi i} = \int_{C_\infty}\frac{dw_a}{2\pi i} - \sum_{j=1}^{a} Res_{w_a=q^2 z_j},$$

where C_0, C_∞ are the small circles around 0, ∞ going anti-clockwise and clockwise direction respectively. Here, for the sake of simplicity, we omit writing the integrands. Then the integral which we consider now is a sum of terms of the form

$$\prod_{d\in D_1}\int_{C_0}\frac{d\xi_d}{2\pi i}\prod_{d\in A_1}\int_{C_\infty}\frac{dw_a}{2\pi i} Res_{w_{a_r}=q^2 z_{j_r}}\cdots Res_{w_{a_1}=q^2 z_{j_1}} Res_{\xi_{d_l}=u_{i_l}}\cdots Res_{\xi_{d_1}=u_{i_1}},$$

where D_1 and A_1 is a subset of $\{a\}$ and $\{d\}$ respectively. Since there is a term $\prod_{a<a'}(1-\frac{w_{a'}}{w_a})\prod_{d<d'}(1-\frac{\xi_{d'}}{\xi_d})$ in the numerator of the integrand, we can assume that $j_{p_1}\neq j_{p_2}(p_1\neq p_2)$, $i_{l_1}\neq i_{l_2}(l_1\neq l_2)$. In $Res_{\xi_{d_l}=u_{i_l}}\cdots Res_{\xi_{d_1}=u_{i_1}}$ the possible poles at $w_a=qu_{i_k}$ are cancelled out with $\prod_a\prod_{l=1}^n(1-\frac{qu_l}{w_a})$. Hence after taking residues in $w_{a_p}s$, there does not appear poles at $u_j=qz_k$. Since there is the term $\prod_d\prod_{j=1}^{n+1}(1-\frac{\xi_d}{q^3z_j})$ in the numerator, the poles at $u_{i_p}=q^3z_{j_k}$ which appear after taking $Res_{w_{a_r}=q^2z_{j_r}}\cdots Res_{w_{a_1}=q^2z_{j_1}}$ are also cancelled out. Finally in the remaining integral $\prod_{d\in D_1}\int_{C_0}\frac{d\xi_d}{2\pi i}\prod_{d\in A_1}\int_{C_\infty}\frac{dw_a}{2\pi i}$ there does not occur pinches of the integral contours at $u_j=qz_k,q^3z_k$. Hence it has no singularities there.
(2): Let us consider the composition

$$\Phi(z_1)\Phi(z_2):V(\Lambda_i)\longrightarrow\hat{V}(\Lambda_{i+1})\otimes(V_1)_{z_1}\otimes(V_1)_{z_2}.$$

By the explicit formula of $<\Lambda_{i+1}|O(\mathbf{z}|\mathbf{u})|\Lambda_i>$ ([2]) $\Phi(z_1)\Phi(z_2)$ is regular at $z_1=q^2z_2$. Since there is no non-zero $U_q'(\hat{s}l_2)$ intertwiner $V(\Lambda_i)\longrightarrow\hat{V}(\Lambda_{i+1})\otimes(V_2)_z$, we have $Pr(z)_{12}\Phi(q^2z_2)\Phi(z_2)=0$. Hence

$$Res_{z_j=q^2z_{j+1}}\frac{1}{f}Pr(z)_{jj+1}\Phi(z_j)\Phi(z_{j+1})=0 \tag{20}$$

for any $1\leq j\leq n$. Using the commutation relations of the vertex operators $\Phi(z)$ and the relations 19, 20

$$\begin{aligned}
&Res_{z_j=q^2z_k}Pr(z)O(\mathbf{z}|\mathbf{u})\\
&=Res_{z_j=q^2z_k}\prod_{l=j}^{k-2}(\frac{z_j}{z_l})^{-1+\overline{i+l}}Pr(z)\check{\bar{R}}(\frac{z_j}{z_{j+1}})^{-1}\cdots\check{\bar{R}}(\frac{z_j}{z_{k-1}})^{-1}O(z_1,\cdots,z_j,z_k,\cdots,z_{n+1}|\mathbf{u})\\
&=\prod_{l=j}^{k-2}(\frac{q^2z_k}{z_l})^{-1+\overline{i+l}}Res_{z_j=q^2z_k}Pr(z)O(z_1,\cdots,z_j,z_k,\cdots,z_{n+1}|\mathbf{u})\\
&=\prod_{l=j}^{k-2}(\frac{q^2z_k}{z_l})^{-1+\overline{i+l}}Pr(z)^{jk}Res_{z_j=q^2z_k}Pr(z)_{jk}O(z_1,\cdots,z_j,z_k,\cdots,z_{n+1}|\mathbf{u})\\
&=0.
\end{aligned}$$

Hence $Pr(z)O(\mathbf{z}|\mathbf{u})$ is regular at $z_j=q^2z_k$ $(j<k)$. We can similarly prove that $Pr(u)O(\mathbf{z}|\mathbf{u})$ is regular at $u_j=q^2u_k$ $(j<k)$. □

Definition 6 (Fused vertex operator)

$$\begin{aligned}
O(z)&=Pr(z)Pr(u)O(\mathbf{z}|\mathbf{u})|_{z_j=q^{-2j+2}z(1\leq j\leq n+1),u_k=q^{-2k+3}z(1\leq k\leq n)}\\
&=\sum_{j,k}v_j^{(n)}\otimes O(z)_{jk}\otimes v_k^{(n+1)}.
\end{aligned}$$

Theorem 4 (1) *The operator $O(z)$ is not zero as a linear map.*

(2) *$O(z)$ is a $U_q'(\hat{s}l_2)$-linear map*

$$V(\Lambda_i) \longrightarrow (V_n)_{q^{-n+2}z} \otimes V(\Lambda_{i+1}) \otimes (V_{n+1})_{q^{-n}z}.$$

Proof
(1): The integral formula of $< \Lambda_{i+1}|O(\mathbf{z}|\mathbf{u})|\Lambda_i >$ implies (see 34, 35)

$$\begin{aligned} &< \Lambda_1|O(z)_{0,n+1}|\Lambda_0 >= (-1)^{[\frac{n}{2}](n-1)}(-q)^{\frac{n(n-2)}{4}-\frac{3}{8}(1-(-1)^n)} \\ &< \Lambda_0|O(z)_{n,0}|\Lambda_1 >= (-1)^{[\frac{n}{2}](n-1)}(-q)^{\frac{n}{12}(8n^2-15n+22)+\frac{3}{8}(1-(-1)^n)}z^{-\frac{n(n-1)}{2}}. \end{aligned}$$

Hence $O(z)$ is not zero as a linear map.
(2): By definition $O(z)$ is $U_q'((\hat{sl}_2)_1)$-linear. Therefore it is sufficient to prove that $O(z)$ commutes with the actions of e_0 and f_0.

Let us prove the commutativity of $O(z)$ with e_0. The case of f_0 is similarly proved. The intertwining properties of $O(\mathbf{z}|\mathbf{u})$ implies

$$\begin{aligned} < v'|O(\mathbf{z}|\mathbf{u})|e_0 v >= \quad & (e_0 \otimes 1) < v'|O(\mathbf{z}|\mathbf{u})|v > +(t_0 \otimes 1) < v'e_0|O(\mathbf{z}|\mathbf{u})|v > \\ & +(t_0 \otimes e_0) < v't_0|O(\mathbf{z}|\mathbf{u})|v > \end{aligned}$$

for any $v \in V(\Lambda_i)$, $v' \in V(\Lambda_{i+1})$. It is sufficient to prove, modulo $\sum N_j \otimes (V_1)_{z_1} \otimes \cdots (V_1)_{z_{n+1}} + (V_1)_{u_1} \otimes \cdots (V_1)_{u_{n+1}} \otimes \sum N_j$, that

$$Pr(u)Pr(z)(e_0 \otimes 1) < v'|O(\mathbf{z}|\mathbf{u})|v >= (e_0 \otimes 1)Pr(u)Pr(z) < v'|O(\mathbf{z}|\mathbf{u})|v >, \qquad (21)$$
$$Pr(u)Pr(z)(t_0 \otimes 1) < v'e_0|O(\mathbf{z}|\mathbf{u})|v >= (t_0 \otimes 1)Pr(u)Pr(z) < v'e_0|O(\mathbf{z}|\mathbf{u})|v >, \quad (22)$$
$$Pr(u)Pr(z)(t_0 \otimes e_0) < v't_0|O(\mathbf{z}|\mathbf{u})|v >= (t_0 \otimes e_0)Pr(u)Pr(z) < v't_0|O(\mathbf{z}|\mathbf{u})|v >, \ (23)$$

at $z_j = q^{-2(j-1)}z(1 \le j \le n+1)$, $u_j = q^{-2(j-1)+1}z(1 \le j \le n)$. It is easily proved, using proposition 5 (2), that the left hand sides of equations 21-23 are regular at $z_j = q^2 z_k (j < k)$, $u_j = q^2 u_k (j < k)$. Hence we can specialize variables as above.

Since t_0 acts on $(V_1)_{u_1} \otimes \cdots \otimes (V_1)_{u_n}$ as t_1^{-1} and $Pr(u)$ is $U_q'((\hat{sl}_2)_1)$ linear, 22 holds. Let us prove the equation 21. According as the decompositions $(V_1)_{u_1} \otimes \cdots \otimes (V_1)_{u_n} \simeq V_n \oplus \sum N_j$, $(V_1)_{z_1} \otimes \cdots \otimes (V_1)_{z_{n+1}} \simeq V_{n+1} \oplus \sum N_j$ as $U_q'((\hat{sl}_2)_1)$ modules, we write

$$\begin{aligned} &< v'|O(\mathbf{z}|\mathbf{u})|v >= (A + A') \otimes (B + B'), \\ &A \in V_n, \quad A' \in \sum N_j, \quad B \in V_{n+1}, \quad B' \in \sum N_j. \end{aligned}$$

Then

$$\begin{aligned} & Pr(u)Pr(z)(e_0 \otimes 1) < v'|O(\mathbf{z}|\mathbf{u})|v > -(e_0 \otimes 1)Pr(u)Pr(z) < v'|O(\mathbf{z}|\mathbf{u})|v > \\ & = (Pr(u)e_0 A - e_0 A) \otimes B + Pr(u)e_0 A' \otimes B. \end{aligned}$$

Since $Pr(u)e_0 A - e_0 A \equiv 0 \bmod \sum N_j$, it is sufficient to prove

$$Pr(u)e_0 A' \otimes B = 0 \text{ at } z_j = q^{-2(j-1)}z(1 \le j \le n+1),\ u_j = q^{-2(j-1)+1}z(1 \le j \le n). \ (24)$$

Lemma 8 *$Pr(u)e_0A'\otimes B$ is regular at $z_j=q^2z_k(j<k)$, $u_j=q^2u_k(j<k)$.*

Proof: Since $Pr(u)(e_0\otimes 1)<v'|O(\mathbf{z}|\mathbf{u})|v>$ is regular at $u_j=q^2u_k(j<k)$ as we already mentioned, $Pr(u)(e_0\otimes 1)(A+A')\otimes B$ is regular at the same place. On the other hand $(A+A')\otimes B$ is regular at $z_j=q^2z_k(j<k)$ and $A\otimes B$ is regular at $z_j=q^2z_k(j<k)$, $u_j=q^2u_k(j<k)$, by proposition 5 (2). Hence $Pr(u)e_0A'\otimes B$ is regular at $z_j=q^2z_k(j<k)$, $u_j=q^2u_k(j<k)$. □

Now let us decompose $<v'|O(\mathbf{z}|\mathbf{u})|v>$ in the following manner:

$$<v'|O(\mathbf{z}|\mathbf{u})|v>=\sum_{j=1}^{n-1}\frac{O_j(\mathbf{z}|\mathbf{u})}{u_j-q^2u_{j+1}}+\tilde{O}(\mathbf{z}|\mathbf{u}), \tag{25}$$

$$O_j(\mathbf{z}|\mathbf{u})=\mathrm{Res}_{u_j=q^2u_{j+1}}(<v'|O(\mathbf{z}|\mathbf{u})|v>-\sum_{k=1}^{j-1}\frac{O_k(\mathbf{z}|\mathbf{u})}{u_k-q^2u_{k+1}})\text{ for }j\geq 2,$$

$$O_1(\mathbf{z}|\mathbf{u})=\mathrm{Res}_{u_1=q^2u_2}<v'|O(\mathbf{z}|\mathbf{u})|v>.$$

Then

Lemma 9 **(1)** $O_j(\mathbf{z}|\mathbf{u})\in\sum N_k\otimes(V_1)_{z_1}\otimes\cdots\otimes(V_1)_{z_{n+1}}$,

(2) *$\tilde{O}(\mathbf{z}|\mathbf{u})$ is regular at $u_j=q^{2(k-j)}u_k(j<k)$,*

(3) *$O_j(\mathbf{z}|\mathbf{u})$ is regular at $u_r=qz_r(1\leq r\leq n)$,*

(4) $O_j(\mathbf{z}|\mathbf{u})|_{u_r=qz_r(1\leq r\leq n)}=0$.

Proof:(1): This follows from 8.
(3): This is obvious from proposition 5 (2).
(4): It follows from

$$\frac{1}{f}|_{u_l=q^2u_{l+1}}=g^{-1}\prod_{j<k}\frac{(\frac{q^4z_k}{z_j})_\infty}{(\frac{q^2z_k}{z_j})_\infty}\prod_{j>k,j,k\neq l,l+1}\frac{(\frac{q^2u_k}{u_j})_\infty}{(\frac{u_k}{u_j})_\infty}\prod_{j\neq l,l+1}\prod_k\frac{(\frac{u_j}{qz_k})_\infty}{(\frac{qu_j}{z_k})_\infty}\frac{\prod_{k=1}^{n+1}(1-\frac{u_{l+1}}{qz_k})}{\prod_{j=l+2}^{n}(1-\frac{u_{l+1}}{u_j})}$$

and 8 that $Res_{u_l=q^2u_{l+1}}<v'|O(\mathbf{z}|\mathbf{u})|v>$ has $\prod_{k=1}^{n+1}(1-\frac{u_{l+1}}{qz_k})$ as a factor of its zero divisor. Taking further residues does not produce poles at $u_{l+1}=qz_k(1\leq k\leq n)$ by proposition 5 (2). Hence $O_j(\mathbf{z}|\mathbf{u})|_{u_r=qz_r(1\leq r\leq n)}=0$.
(2): Let us prove, for $2\leq j\leq n$, that

$$<v'|O(\mathbf{z}|\mathbf{u})|v>-\sum_{r=1}^{j-1}\frac{O_r(\mathbf{z}|\mathbf{u})}{u_r-q^2u_{r+1}}\text{ is regular at }u_l=q^{2(s-l)}u_s(l<s,1\leq l\leq j-1)$$

by the induction on j. The $j=2$ case is obvious from proposition 5 (2).

Suppose that the statement is true for $1\leq j\leq k$. We have

$$<v'|O(\mathbf{z}|\mathbf{u})|v>-\sum_{r=1}^{k}\frac{O_r(\mathbf{z}|\mathbf{u})}{u_r-q^2u_{r+1}}=O^{(1)}(\mathbf{z}|\mathbf{u})-O_k(\mathbf{z}|\mathbf{u}),$$

$$O^{(1)}(\mathbf{z}|\mathbf{u})=<v'|O(\mathbf{z}|\mathbf{u})|v>-\sum_{r=1}^{k-1}\frac{O_r(\mathbf{z}|\mathbf{u})}{u_r-q^2u_{r+1}},$$

$$O_k(\mathbf{z}|\mathbf{u})=Res_{u_k=q^2u_{k+1}}O^{(1)}(\mathbf{z}|\mathbf{u}).$$

By the induction hypothesis $O^{(1)}(\mathbf{z}|\mathbf{u})$ is regular at $u_l = q^{2(s-l)}u_s (l < s, 1 \le l \le k-1)$. Hence $O_k(\mathbf{z}|\mathbf{u})$ and consequently $O^{(1)}(\mathbf{z}|\mathbf{u}) - O_k(\mathbf{z}|\mathbf{u})$ are regular at $u_l = q^{2(s-l)}u_s (l < s, 1 \le l \le k-1)$. The definition of a residue and proposition 5 (2) imply that $O^{(1)}(\mathbf{z}|\mathbf{u}) - O_k(\mathbf{z}|\mathbf{u})$ is regular at $u_k = q^{2(s-k)}u_s (k < s)$. Hence the statement is proved for $j = k+1$. □

Using the decomposition 25 we have

$$\begin{aligned} Pr(u)e_0 A' \otimes B = \quad & \sum_{j=1}^{n-1} \frac{1}{u_j - q^2 u_{j+1}} Pr(u)Pr(z)(e_0 \otimes 1)O_j(\mathbf{z}|\mathbf{u}) \\ & + Pr(u)Pr(z)(e_0 \otimes 1)(1 - Pr(u))\tilde{O}(\mathbf{z}|\mathbf{u}). \end{aligned}$$

Note that, in $(V_1)_{u_1} \otimes (V_1)_{u_2}$

$$e_0 w = (u_1 - q^2 u_2) v_1^{(1)} \otimes v_1^{(1)}.$$

Since $\tilde{O}(\mathbf{z}|\mathbf{u})$ has no poles at $u_j = q^{2(s-j)}u_s (j < s)$ we can conclude that

$$Pr(u)Pr(z)(e_0 \otimes 1)(1 - Pr(u))\tilde{O}(\mathbf{z}|\mathbf{u})|_{u_j = q^{-2(j-1)+1}z} = 0.$$

Since each $O_j(\mathbf{z}|\mathbf{u})$ has a zero divisor of the form $\prod_{j=1}^{n+1}(1 - \frac{u_l}{qz_j})$

$$\sum_{j=1}^{n-1} \frac{1}{u_j - q^2 u_{j+1}} Pr(u)Pr(z)(e_0 \otimes 1)O_j(\mathbf{z}|\mathbf{u})|_{u_j = qz_j (1 \le j \le n)} = 0.$$

Taking into account that $Pr(u)e_0 A' \otimes B$ has no pole at all we can conclude that

$$Pr(u)e_0 A' \otimes B|_{z_j = q^{-2(j-1)}z (1 \le j \le n+1), u_j = q^{-2(j-1)}z (1 \le j \le n)} = 0.$$

Hence 21 is proved. The equation 23 is similarly proved. □

Corollary 3

$$\begin{aligned} {}^{V_n}\Phi^{V_{n+1}}(z) &= (-1)^{[\frac{n}{2}](n-1)}(-q)^{-\frac{n(n-2)}{4}+\frac{3}{8}(1-(-1)^n)} O(q^n z) \ \textit{on}\ V(\Lambda_0) \\ &= (-1)^{[\frac{n}{2}]}(-q)^{-\frac{n}{12}(2n^2-9n+10)-\frac{3}{8}(1-(-1)^n)} z^{\frac{n(n-1)}{2}} O(q^n z) \ \textit{on}\ V(\Lambda_1). \end{aligned}$$

7 Commutation relations of vertex operators

Using the fusion construction we can determine the commutation relations of vertex operators in section 4 and 6. Here we give only commutation relations which is used for the application to the vertex models.

Theorem 5 **(1)**

$$
\begin{aligned}
&{}^{V_n}\Phi_{V_{n+1}}(z)_{V_n}\Phi^{V_{n+1}}(z)=c_i id_{(V_n)_z\otimes V(\Lambda_i)},\\
&c_i=(-1)^{[\frac{n}{2}]n+\frac{1}{6}n(n-1)(n+4)}q^{-\frac{1}{6}n(n^2-7)+in(n-2)}\\
&\qquad [n+1]^{1-i}z^{\frac{n(n-1)}{2}}\frac{(q^2)_\infty(q^4)_\infty}{(q^{2n+2})_\infty^2(q^2;q^2)_n},
\end{aligned}
$$

where $(z;p)_n=\prod_{l=0}^{n-1}(1-zp^l)$.

(2)

$$
\begin{aligned}
&(-1)^n q^{-3[\frac{n+1-i}{2}]+n(i+1)}(\frac{z}{w})^{i-\frac{1}{2}}\check{R}_{n+1,1}(\frac{z}{w})O(q^nz)\Phi(w)=\Phi(w)O(q^nz),\\
&\check{R}_{n+1,1}(z)=z^{-\frac{1}{2}}r_{n+1}(z)\breve{\check{R}}_{n+1,1}(z),\quad r_{n+1}(z)=\frac{(q^{n+2}z)_\infty(q^{n+4}z^{-1})_\infty}{(q^{n+2}z^{-1})_\infty(q^{n+4}z)_\infty}
\end{aligned}
$$

Proposition 6

$$
P_F^{n+1}O(q^{-2}z)O(z)=(-1)^{n+i}q^{\frac{n(n+1)}{2}+i}\frac{(q^2)_\infty(q^4)_\infty}{(q^{2n+2})_\infty^2(q^2;q^2)_n}w_n\otimes id_{V(\Lambda_i)},
$$

where w_n is the highest weight vector with weight zero in $V_n\otimes V_n$ which is explicitly described in 18 and 17.

We shall first prove Proposition 6 and after that deduce (1) of Theorem 5. Let us set

$$
\begin{aligned}
&\tilde{O}(\mathbf{z}',\mathbf{z}|\mathbf{u}',\mathbf{u})=\\
&\Phi^{V^{*a}}(z_1')\Phi(z_1)\cdots\Phi^{V^{*a}}(z_{n+1}')\Phi(z_{n+1})\Psi(q^{-2}u_n')\Psi^{V^{*a^{-1}}}(q^{-2}u_n)\cdots\Psi(q^{-2}u_1')\Psi^{V^{*a^{-1}}}(q^{-2}u_1).
\end{aligned}
$$

Using the commutation relations of the vertex operators $\Phi(z)$ and $\Psi(z)$, we have

$$
\begin{aligned}
&\frac{h}{ff'}(-1)^{\frac{n(n-1)}{2}+i}q^{n+i}\prod_{j<k}(\frac{z_j}{z_k'})^{\overline{i+j-k}}\prod_{j<k}(\frac{u_j'}{u_k})^{\overline{i+j-k-1}}\prod_{j,k}(\frac{u_j'}{q^2z_k})^{\overline{i+j-k-1}}\\
&\breve{\check{R}}(\frac{u_1'}{q^2u_n})\cdots\breve{\check{R}}(\frac{u_1'}{q^2u_2})\cdots\breve{\check{R}}(\frac{u_{n-1}'}{q^2u_n})\breve{\check{R}}(\frac{q^2z_1}{z_{n+1}'})\cdots\breve{\check{R}}(\frac{q^2z_1}{z_2'})\cdots\breve{\check{R}}(\frac{q^2z_n}{z_{n+1}'})\tilde{O}(\mathbf{z}',\mathbf{z}|\mathbf{u}',\mathbf{u})\\
&=O(q^{-2}\mathbf{z}'|q^{-2}\mathbf{u}')O(\mathbf{z}|\mathbf{u}),
\end{aligned}
\tag{26}
$$

$$
g=\prod_{j<k}r(\frac{q^2z_j}{z_k'})\prod_{j<k}r(\frac{u_j'}{q^2u_k})\prod_{j,k}\frac{\theta_{q^4}(\frac{q^3z_j}{u_k'})}{\theta_{q^4}(\frac{u_k'}{qz_j})}
$$

where $f'=f(\mathbf{z}'|\mathbf{u}')$, $q^{-2}\mathbf{z}'=(q^{-2}z_1',\cdots,q^{-2}z_{n+1}')$ etc. and $\overline{i+j-k}$ etc. means that the number $i+j-k$ should be considered modulo two, inparticular $\overline{i+j-k}=0$ or 1. Note that

$$
h|_{u_j=q^3z_{j+1},u_j'=q^3z_{j+1}'(1\le j\le n)}=\prod_{j=2}^{n+1}(1-\frac{z_j}{z_j'})\tilde{h},
$$

$$
\tilde{h}=q^{-n(n+1)}\prod_{2\le j<k\le n+1}\frac{(\frac{q^4z_j}{z_k'})_\infty^2(\frac{q^4z_j'}{z_k})_\infty(\frac{z_j'}{z_k})_\infty(1-\frac{z_k}{z_j'})}{(\frac{q^2z_j}{z_k'})_\infty(\frac{q^6z_j}{z_k'})_\infty(\frac{z_j'}{q^2z_k})_\infty(\frac{q^2z_j'}{z_k})_\infty}\prod_{j=2}^{n+1}\frac{(\frac{q^4z_1}{z_j'})_\infty^2(\frac{q^4z_j}{z_j'})_\infty(\frac{q^4z_j'}{z_j})_\infty}{(\frac{q^6z_1}{z_j'})_\infty(\frac{q^2z_1}{z_j'})_\infty(\frac{q^2z_j'}{z_j})_\infty(\frac{q^2z_j}{z_j'})_\infty}.
$$

Specializing the variables to $u_j = q^3 z_{j+1}, u'_j = q^3 z'_{j+1} (1 \le j \le n)$ in both hand sides of the equation 26, after that setting $z_j = z'_j (1 \le j \le n+1)$ and using 7

$$\lim_{z_j \to z'_j} (1 - \frac{z_j}{z'_j}) \Psi(z'_j) \Psi^{V^{*a^{-1}}}(z_j) = -gw \otimes \mathrm{id}_{V(\Lambda_i)}$$

we have

$$\frac{\tilde{\tilde{h}}}{\tilde{f}^2} (-1)^{\frac{n(n+1)}{2}+i} q^{n+i} g^n \prod_{1 \le j < k \le n+1} (\frac{z_j}{z_k})^{\overline{i+j-k}} \prod_{2 \le j < k \le n+1} (\frac{z_j}{z_k})^{\overline{i+j-k-1}} \prod_{j=2}^{n+1} \prod_{k=1}^{n+1} (\frac{q z_j}{z_k})^{\overline{i+j-k}}$$
$$\prod_{2 \le j < k \le n+1} \frac{1}{1 - \frac{z_j}{q^2 z_k}} R_n(z) w^{\otimes n} \otimes \tilde{R}_{n+1}(z) \Phi^{V^{*a}}(z_1) \Phi(z_1) \cdots \Phi^{V^{*a}}(z_{n+1}) \Phi(z_{n+1})$$
$$O(q^{-2}\mathbf{z} | q z_{n+1}, \cdots, q z_2) O(\mathbf{z} | q^3 z_{n+1}, \cdots, q^3 z_2), \tag{27}$$

where

$$R_n(\mathbf{z}) = \check{\bar{R}}(\frac{z_2}{q^2 z_{n+1}}) \cdots \check{\bar{R}}(\frac{z_2}{q^2 z_3}) \cdots \check{\bar{R}}(\frac{z_n}{q^2 z_{n+1}})$$
$$\tilde{R}_{n+1}(\mathbf{z}) = \check{\bar{R}}(\frac{q^2 z_1}{z_{n+1}}) \cdots \check{\bar{R}}(\frac{q^2 z_1}{z_2}) \cdots \check{\bar{R}}(\frac{q^2 z_n}{z_{n+1}}),$$
$$\tilde{\tilde{h}} = q^{-n(n+1)} g^{-2n} \prod_{2 \le j < k \le n+1} \frac{(\frac{q^4 z_j}{z_k})^3_\infty (\frac{z_j}{z_k})_\infty (1 - \frac{z_k}{z_j})}{(\frac{q^2 z_j}{z_k})^3_\infty (\frac{q^6 z_j}{z_k})_\infty} \prod_{j=2}^{n+1} \frac{(\frac{q^4 z_1}{z_j})^2_\infty}{(\frac{q^6 z_1}{z_j})_\infty (\frac{q^2 z_1}{z_j})_\infty},$$
$$\tilde{f} = g^{-n} \prod_{2 \le j < k \le n+1} \frac{(\frac{z_j}{z_k})_\infty (\frac{q^4 z_j}{z_k})_\infty}{(\frac{q^2 z_j}{z_k})^2_\infty},$$

and $w = v_0^{(1)} \otimes v_1^{(1)} - q v_1^{(1)} \otimes v_0^{(1)}$.

Lemma 10 *Let Pr_n be the $U'_q((\hat{sl}_2)_1)$ linear projection $V_1^{\otimes n} \otimes V_1^{\otimes n} \longrightarrow V_n \otimes V_n$. Then we have*

$$Pr_n R_n(\mathbf{z}) w^{\otimes n} = q^{\frac{n(n-1)}{2}} \prod_{2 \le j < k \le n+1} \frac{1 - \frac{z_j}{q^2 z_k}}{1 - \frac{z_j}{z_k}} w_n.$$

Proof: Since $Pr_n R_n(\mathbf{z}) w^{\otimes n}$ belongs to the trivial representation of $V_n \otimes V_n$, we have $Pr_n R_n(\mathbf{z}) w^{\otimes n} = c w_n$ for some scalar function c. The function c is the coefficient of $v_0^{(n)} \otimes v_n^{(n)}$ in the right hand side. Let us calculate the coefficient of $v_0^{(1)\otimes n} \otimes v_1^{(1)\otimes n}$ in $R_n(\mathbf{z}) w^{\otimes n}$. It is easy to see that this coefficient is the same as that of $v_0^{(1)\otimes n} \otimes v_1^{(1)\otimes n}$ in $R_n(\mathbf{z})(v_0^{(1)} \otimes v_1^{(1)})^{\otimes n}$. The latter coefficient is easily calculated and coinsides with the function in the statement of the lemma. □

Let $(P_F^1)^{\otimes(n+1)}$ be the $U'_q(\hat{sl}_2)$ linear map $(V_1)^{a*}_{z_1} \otimes (V_1)_{z_1} \otimes \cdots \otimes (V_1)^{a*}_{z_{n+1}} \otimes (V_1)_{z_{n+1}} \longrightarrow F$ defined by $(P_F^1)^{\otimes(n+1)}(\otimes_{l=1^{n+1}} (v_{j_l}^{(1)*} \otimes v_{k_l}^{(1)})) = \prod_{l=1}^{n+1} \delta_{j_l,k_l}$ and P_F^{n+1} the dual pairing map $(V_{n+1})^{*a}_{q^{-n}z} \otimes (V_{n+1})_{q^{-n}z} \longrightarrow F$.

Lemma 11 *There is an equation*

$$cP_F^{n+1}Pr_{n+1}\tilde{R}_{n+1}(\mathbf{z}) = (P_F^1)^{\otimes(n+1)},$$
$$c = q^{-\frac{n(n+1)}{2}}\frac{(q^2;q^2)_{n+1}}{(1-q^2)^{n+1}}$$

at $z_j = q^{-2(j-1)}z(1 \le j \le n+1)$.

Note that the R-matrix $\bar{R}(\frac{q^2 z_j}{z_k})(j<k)$ is regular at $\frac{z_j}{z_k} = q^{2(k-j)}$ and $\bar{R}(\frac{q^2 z_j}{z_k})^{-1} = \bar{R}(\frac{z_k}{q^2 z_j})$ which is also regular at $\frac{z_j}{z_k} = q^{2(k-j)}$. Hence there exists the inverse of $\tilde{R}_{n+1}(\mathbf{z})$ which is regular at $z_j = q^{-2(j-1)}(1 \le j \le n+1)$. Let us set $\varphi(\mathbf{z}) = (P_F^1)^{\otimes(n+1)}\tilde{R}_{n+1}^{-1}(\mathbf{z})$,

$$\begin{array}{ccc}
(V_1)^{*a}_{z_1} \otimes (V_1)_{z_1} \otimes \cdots \otimes (V_1)^{*a}_{z_{n+1}} \otimes (V_1)_{z_{n+1}} & \stackrel{(P_F^1)^{\otimes(n+1)}}{\longrightarrow} & F \\
\downarrow \tilde{R}_{n+1}(\mathbf{z}) & & \downarrow \mathrm{id} \\
(V_1)^{*a}_{z_1} \otimes \cdots (V_1)^{*a}_{z_{n+1}} \otimes (V_1)_{z_1} \otimes \cdots \otimes (V_1)_{z_{n+1}} & \stackrel{\varphi(\mathbf{z})}{\longrightarrow} & F.
\end{array}$$

Then

sublemma 1

$$\varphi(\mathbf{z})(N_j \otimes V_z \otimes \cdots \otimes V_{q^{-2n}z}) = \varphi(\mathbf{z})(V_z^{*a} \otimes \cdots \otimes V^{*a}_{q^{-2n}z} \otimes N_j) = 0$$

for all $1 \le j \le n+1$.

Proof: Since $\varphi(\mathbf{z})$ is $U_q'(\hat{s}l_2)$ linear map we have

$$\begin{aligned}
&\varphi(\mathbf{z})(v_{j_1}^{(1)*} \otimes \cdots \otimes v_{j_{n+1}}^{(1)*} \otimes v_{k_1}^{(1)} \otimes \cdots \otimes v_{k_{n+1}}^{(1)}) \\
&= \beta < v_{j_{n+1}}^{(1)*} \otimes \cdots \otimes v_{j_1}^{(1)*}, \tilde{\tilde{R}}_{n+1}(z)(v_{k_1}^{(1)} \otimes \cdots \otimes v_{k_{n+1}}^{(1)}) >,
\end{aligned} \tag{28}$$

for some scalar function β. Here $\tilde{\tilde{R}}_{n+1}(z)$ is the specialization of the variables to $z_j = q^{-2(j-1)}z(1 \le j \le n+1)$ of the $U_q'(\hat{s}l_2)$ intertwiner $(V_1)_{z_1} \otimes \cdots \otimes (V_1)_{z_{n+1}} \longrightarrow (V_1)_{z_{n+1}} \otimes \cdots \otimes (V_1)_{z_1}$ normalized as $\tilde{\tilde{R}}_{n+1}(\mathbf{z})(v_0^{(1)\otimes n}) = v_0^{(1)\otimes n}$. In fact, for generic values of $z_j's$ for which $(V_1)^{*a}_{z_1} \otimes \cdots \otimes (V_1)^{*a}_{z_{n+1}} \otimes (V_1)_{z_1} \otimes \cdots \otimes (V_1)_{z_{n+1}}$ is irreducible, the $U_q'(\hat{s}l_2)$ linear map $(V_1)^{*a}_{z_1} \otimes \cdots \otimes (V_1)^{*a}_{z_{n+1}} \otimes (V_1)_{z_1} \otimes \cdots \otimes (V_1)_{z_{n+1}} \longrightarrow F$ is unique up to scalar factor and given by $\tilde{\tilde{R}}_{n+1}(\mathbf{z})$ as in the right hand side of 28. Since $\beta = \varphi(\mathbf{z})(v_0^{(1)*\otimes(n+1)} \otimes v_0^{(1)\otimes(n+1)})$ and $\tilde{R}_{n+1}(\mathbf{z})$ is regular at $z_j = q^{2(k-j)}z_k(j<k)$, β is also regular at $z_j = q^{2(k-j)}z_k(j<k)$. Hence 28 holds at $z_j = q^{-2(j-1)}z(1 \le j \le n+1)$. By Proposition 3 we have $\tilde{\tilde{R}}_{n+1}(\mathbf{z})(N_j) = 0$ and hence $\varphi(\mathbf{z})(V_z^{*a} \otimes \cdots \otimes V^{*a}_{q^{-2n}z} \otimes N_j) = 0$.

Let us prove the remaining equation. Note that the base of the trivial representation in $V_u^{*a} \otimes V^{*a}_{q^{-2}u}$ is given by $v_1^{(1)*} \otimes v_0^{(1)*} - qv_0^{(1)*} \otimes v_1^{(1)*}$. Taking into account the fact that, in the left part of the right hand side of the equality 28, the order of the tensor product is reversed, we set $w^* = v_0^{(1)*} \otimes v_1^{(1)*} - qv_1^{(1)*} \otimes v_0^{(1)*}$. Then, by calculations, we have

$$< w^*, f_1^k(v_0^{(1)} \otimes v_0^{(1)}) >= 0 \text{ for } 0 \le k \le 2.$$

Since, by Proposition 3, $Im\tilde{\tilde{R}}_{n+1}(\mathbf{z}) \simeq (V_{n+1})_{q^{-n}z}$ which is generated by $v_0^{(1)\otimes(n+1)}$ over $U_q'((\hat{sl}_2)_1)$, we have

$$\varphi(\mathbf{z})(N_j \otimes V_z \otimes \cdots \otimes V_{q^{-2n}z}) = 0.$$

□

Let us continue the proof of lemma. By the sublemma the map $\varphi(\mathbf{z})$ induces the $U_q'(\hat{s}l_2)$ linear map

$$(V_{n+1})^{*a}_{q^{-n}z} \otimes (V_{n+1})_{q^{-n}z} \longrightarrow F$$

which we denote the same symbol. Hence $\varphi(\mathbf{z})$ is a scalar multiple of the canonical pairing map P_F^{n+1}, $\varphi(\mathbf{z}) = cP_F^{n+1}$.

Let us determine the scalar c. Note that $c = \varphi(\mathbf{z})((v_0^{(1)})^{\otimes(n+1)} \otimes (v_1^{(1)})^{\otimes(n+1)})$. It is easily proved that

$$\varphi(\mathbf{z})((v_0^{(1)})^{\otimes(n+1)} \otimes (v_1^{(1)})^{\otimes(n+1)}) =< (v_0^{(1)*} \otimes v_1^{(1)*})^{\otimes(n+1)}, \tilde{R}_{n+1}^{-1}(z)(v_0^{(1)})^{\otimes(n+1)} \otimes (v_1^{(1)})^{\otimes(n+1)}) > .$$

Recall that

$$\tilde{R}^{-1}_{n+1}(z) = \check{\tilde{R}}(\frac{z_{n+1}}{q^2 z_n}) \cdots \check{\tilde{R}}(\frac{z_2}{q^2 z_1}) \cdots \check{\tilde{R}}(\frac{z_{n+1}}{q^2 z_1})$$

with $z_j = q^{-2(j-1)}z (1 \le j \le n+1)$.

It follows from those description we have

$$c = q^{\frac{n(n+1)}{2}} \prod_{1\le j<k\le n+1} \frac{1-\frac{z_k}{q^2 z_j}}{1-\frac{z_k}{z_j}} = q^{\frac{n(n+1)}{2}} \frac{\prod_{l=1}^{n+1}(1-q^{-2l})}{(1-q^{-2})^{n+1}}.$$

□

Proof of Proposition 6: Taking $(1 \otimes P_F^{n+1})(Pr_n \otimes Pr_{n+1})$ in both hand sides of the equation 27 and using Lemma 10, Lemma 11, equation 6, we have the equality in the statement of Proposition 6. □

Now (1) of Theorem 5 is derived from Proposition 6 in the following manner. Let us introduce the $U_q'(\hat{s}l_2)$ intertwiners

$$\begin{aligned} {}^{V_n}\Phi^{V^{*a}_{n+1}}(z) &: V(\Lambda_i) \longrightarrow (V_n)_z \otimes V(\Lambda_{i+1}) \otimes (V_{n+1})^{*a}_z, \\ {}^{V_n^{*a^{-1}}}\Phi^{V_{n+1}}(z) &: V(\Lambda_i) \longrightarrow (V_n)^{*a^{-1}}_z \otimes V(\Lambda_{i+1}) \otimes (V_{n+1})_z, \end{aligned}$$

by

$$\begin{aligned} < v_j^{(n+1)}, {}^{V_n}\Phi^{V^{*a}_{n+1}}(z)u > &= {}^{V_n}\Phi_{V_{n+1}}(z)(u \otimes v_j^{(n+1)}), \\ < v_j^{(n)}, {}^{V_n^{*a^{-1}}}\Phi^{V_{n+1}}(z)u > &= {}_{V_n}\Phi^{V_{n+1}}(z)(v_j^{(n)} \otimes u). \end{aligned}$$

Then

$$
\begin{aligned}
&{}^{V_n}\Phi^{V^{*a}_{n+1}}(z) = (-1)^{n+1-i} q^{-(n+1)(1-i)} [n+1]^{i\, V_n}\Phi^{V_{n+1}}(q^{-2}z), \\
&{}^{V_n^{*a^{-1}}}\Phi^{V_{n+1}}(z) = {}^{V_n}\Phi^{V_{n+1}}(z).
\end{aligned}
$$

Those equations implies that the equation

$$
{}^{V_n}\Phi_{V_{n+1}}(z)_{V_n}\Phi^{V_{n+1}}(z) = \gamma \mathrm{id}_{(V_n)_z \otimes V(\Lambda_i)}
$$

is equivalent to

$$
P_F^{n+1\, V_n}\Phi^{V_{n+1}}(q^{-2}z)^{V_n}\Phi^{V_{n+1}}(z) = (-1)^{n+i} q^{i(n+1)} [n+1]^{i-1} \gamma \sum_{j=0}^{n} v_j^{(n)*} \otimes v_j^{(n)} \otimes \mathrm{id}_{V(\Lambda_i)}.
$$

Rewriting these equations in terms of $O(z)$, we have

$$
\begin{aligned}
P_F^{n+1} O(q^{n-2}z) O(q^n z) &= \gamma (-1)^{[\frac{n}{2}]n+n+i+\frac{1}{6}n(n-1)(n+4)} q^{\frac{1}{6}n(n-1)(n+4)+i(-n^2+2n+1)} \\
&\quad z^{-\frac{n(n-1)}{2}} [n+1]^{-1+i} (\sum_{j=0}^{n} v_j^{(n)*} \otimes v_j^{(n)}) \otimes \mathrm{id}_{V(\Lambda_i)}.
\end{aligned}
$$

Theorem 5 (1) follows from this equation and Proposition 6. □

Proof of Theorem 5 (2): Using the commutaion relations of $\Phi(z)$ and $\Psi(z)$ we have

$$
\begin{aligned}
&(-1)^{n+1} \prod_{l=1}^{n} (\frac{u_l}{w})^{-\overline{i+l}} \prod_{j=1}^{n+1} (\frac{z_j}{w})^{\overline{i+l-1}} \prod_{j=1}^{n+1} r(\frac{z_j}{w}) \prod_{l=1}^{n} \frac{\theta_{q^4}(\frac{qu_l}{w})}{\theta_{q^4}(\frac{qw}{u_l})} \check{\bar{R}}(\frac{z_1}{w}) \cdots \check{\bar{R}}(\frac{z_{n+1}}{w}) O(\mathbf{z}|\mathbf{u}) \Phi(w) \\
&= \Phi(w) O(\mathbf{z}|\mathbf{u}).
\end{aligned} \tag{29}
$$

Similarly to the Proposition 5, we can prove that $(Pr(z) \otimes Pr(u)) O(\mathbf{z}|\mathbf{u}) \Phi(w)$ and $(Pr(z) \otimes Pr(u)) \Phi(w) O(\mathbf{z}|\mathbf{u})$ give well-difined $U_q^{'}(\hat{s}l_2)$-intertwinwers at $z_j = q^{-2j+2}z$, $u_j = q^{-2j+3}z$. Hence, by Theorem 3, we have

$$
(Pr(z) \otimes Pr(u)) \Big[\check{\bar{R}}(\frac{z_1}{w}) \cdots \check{\bar{R}}(\frac{z_{n+1}}{w}) O(\mathbf{z}|\mathbf{u}) \Phi(w) \Big]_{z_j = q^{-2j+2}z, u_j = q^{-2j+3}z} \tag{30}
$$

$$
= c(z,w) \check{\bar{R}}_{n+1,1}(\frac{q^{-n}z}{w}) \Big[(Pr(z) \otimes Pr(u)) O(\mathbf{z}|\mathbf{u}) \Phi(w) \Big]_{z_j = q^{-2j+2}z, u_j = q^{-2j+3}z} \tag{31}
$$

for some scalar function $c(z,w)$. Comparing the coefiicient of $v_0^{(1)} \otimes v_0^{(n)}$ we conclude that $c(z,w) \equiv 1$. Taking $Pr(z) \otimes Pr(u)$ of the both hand sides of the equation 29 and substituting $z_j = q^{-2(j-1)}z (1 \le j \le n+1)$, $u_j = q^{-2j+3}z (1 \le j \le n)$, we obtain the desired equation. □

8 Inhomogeneous vertex model of 6-vertex type

In this section we denote $(V_s)_1$ by V_s for the sake of simplicity and assume $-1 < q < 0$. Let us consider the two dimensional regular square infinite lattice. Fix the positive integer N and non-negative integers $s_1, \cdots, s_N$ and vertical lines $l_1, \cdots, l_N$. Then the vertex model which we study here is defined in the following way. We associate the representation V_1 of $U_q'(\hat{s}l_2)$ with each edge on horizontal lines and on vertical lines except $l_1, \cdots, l_N$. With each edge on the line l_j we associate the vector space V_{s_j}. For each vertex the Boltzmann weight is given by the corresponding R-matrix, $R_{11}(1)$, $R_{s_j 1}(1)$. We can assume that the lines $l_1, \cdots, l_N$ are successive by including 1 in the set of s_j. Let us first give the mathematical objects and after that explain the validity of them. We use the following vertex operators in this section.

Definition 7 *The intertwiners*

$$
\begin{aligned}
{}^nO^{n+1}(z) &: V(\Lambda_i) \longrightarrow (V_n)_{q^2 z} \otimes V(\Lambda_{i+1}) \otimes (V_{n+1})_z, \\
{}_nO^{n+1}(z) &: (V_n)_z \otimes V(\Lambda_i) \longrightarrow V(\Lambda_{i+1}) \otimes (V_{n+1})_z, \\
{}^nO_{n+1}(z) &: V(\Lambda_i) \otimes (V_{n+1})_z \longrightarrow (V_n)_z \otimes V(\Lambda_{i+1}),
\end{aligned}
$$

are defined by

$$
\begin{aligned}
&{}^nO^{n+1}(z) = O(q^n z), \quad {}_nO^{n+1}(z)(v_j^{(n)} \otimes \cdot) = < v_j^{(n)}, {}^nO^{n+1}(z) >, \\
&{}^nO_{n+1}(z)(\cdot \otimes v_j^{(n+1)}) = < v_j^{(n+1)}, {}^nO^{n+1}(q^{-2}z) >,
\end{aligned}
$$

where the pairing is defined by the isomorphism $(V_n)_{q^2 z} \simeq (V_n)_z^{*a^{-1}}$, $(V_{n+1})_{q^{-2}z} \simeq (V_{n+1})_z^{*a}$.

Theorem 5 (2) and Proposition 6 imply the commutation relations

$$
(-1)^n q^{3[\frac{n+1-i}{2}]-n(i+1)} (\frac{w}{z})^{i-\frac{1}{2}} \check{R}_{1,n+1}(\frac{w}{z}) \Phi(w) {}_nO^{n+1}(z) = {}_nO^{n+1}(z)\Phi(w) \tag{32}
$$

$$
{}^nO_{n+1}(z) {}_nO^{n+1}(z) = (-1)^{n+i} q^{\frac{n(n+1)}{2}+i} \frac{(q^2)_\infty (q^4)_\infty}{(q^{2n+2})_\infty^2 (q^2 : q^2)_n} \mathrm{id}_{(V_n)_z \otimes V(\Lambda_i)}.
$$

on $V(\Lambda_i)$. The representation theoretical formulation for the model is given by

(Space) The space acted by the transfer matrix is

$$
\begin{aligned}
&\mathcal{H} = \oplus_{i,j=0,1} \mathcal{H}_{s_N \cdots s_1, ij}, \\
&\mathcal{H}_{s_N \cdots s_1, ij} = V_{s_N - 1} \otimes \cdots \otimes V_{s_1 - 1} \otimes V(\Lambda_i) \otimes V(\Lambda_j)^{*a}
\end{aligned}
$$

(Transfer matrix) The transfer matrix is given by

$$
T(z) = \mathrm{id} \otimes T_{XXZ}(z),
$$

where $T_{XXZ}(z)$ is the transfer matrix of the 6-vertex model actiong on $\oplus_{i,j=0,1} V(\Lambda_i) \otimes V(\Lambda_j)^{*a}$. Explicitly $T_{XXZ}(z) = g\Phi^*(z)\Phi(z)$ and $\Phi^*(z) = {}^t\Phi(q^2 z) : (V_1)_z \otimes V(\Lambda_i)^{*a} \longrightarrow V(\Lambda_{i+1})^{*a}$, where $g = \frac{(q^2)_\infty}{(q^4)_\infty}$.

(Ground state) The space of vacuum vectors V_{vac} is

$$V_{vac}=\oplus_{i,j=0,1}V_{s_N-1}\otimes\cdots\otimes V_{s_1-1}\otimes F|vac>_{XXZ,i},$$

where $|vac>_{XXZ,i}$ is the vacuum vector of the XXZ-model in $V(\Lambda_i)\otimes V(\Lambda_i)^{*a}$.

(Excited states) The creation and annihilation operators are given by

$$\varphi_j^*(z)=1\otimes\varphi^*_{j,XXZ}(z),\quad \varphi_j(z)=1\otimes\varphi_{j,XXZ}(z),$$

where $\varphi^*_{j,XXZ}(z)$, $\varphi_{j,XXZ}(z)$ are the creation and annihilation operators of the XXZ model,

$$\varphi^*_{j,XXZ}(z)=<v_j^{(1)},\Psi^{V^{*a^{-1}}}(z)>,\quad \varphi_{j,XXZ}(z)=<v_j^{(1)*},\Psi(z)>.$$

(Local operators) For $L\in\mathrm{End}(V_{s_N}\otimes\cdots\otimes V_{s_1})$ the corresponding local operator $\mathcal{L}$ is defined by

$$\begin{aligned}
&\mathcal{L}=\Phi^{s_N,\cdots,s_1}(1,\cdots,1)^{-1}(1\otimes L)\Phi^{s_N,\cdots,s_1}(1,\cdots,1),\\
&\Phi^{s_N,\cdots,s_1}(z_N,\cdots,z_1)={}_{s_N-1}O^{s_N}(z_N)\cdots{}_{s_1-1}O^{s_1}(z_1),\\
&\Phi^{s_N,\cdots,s_1}(z_N,,\cdots,z_1)^{-1}=c_{i,N}{}^{s_1-1}O_{s_1}(z_1)\cdots{}^{s_N-1}O_{s_N}(z_N)\text{ on }\mathcal{H}_{s_N\cdots s_1,ij},\\
&c_{i,N}=(-1)^{\sum_{j=1}^N s_j+iN+\frac{N(N-3)}{2}}q^{-\sum_{j=1}^N\frac{s_j(s_j-1)}{2}-\sum_{j=0}^{N-1}\overline{i+j}}\frac{\prod_{j=1}^N(q^{2s_j})^2_\infty(q^2;q^2)_{s_j-1}}{(q^2)^N_\infty(q^4)^N_\infty}.
\end{aligned}$$

(Correlation functions) The expectation values of the local operator $\mathcal{L}$ is given by

$$<\mathcal{L}>_i=\frac{\mathrm{tr}_{V_{s_N-1}\otimes\cdots\otimes V_{s_1-1}\otimes V(\Lambda_i)}((1\otimes q^{-2\rho})\mathcal{L})}{s_1\cdots s_N\mathrm{tr}_{V(\Lambda_i)}(q^{-2\rho})},$$

where $\rho=\Lambda_0+\Lambda_1$ and 1 is the identity operator acting on $V_{s_N-1}\otimes\cdots\otimes V_{s_1-1}$.

Let us explain why we have given the mathematical setting as above. The less obvious definition is that of the transfer matrix. If it is permitted then others are rather natural compared with the case of the XXZ model. So we shall explain the reason of our definition of the transfer matrix. The natural definition of the transfer matrix $T(z)$ should be

$$\begin{aligned}
&T(z):V(\Lambda_i)\otimes V_{s_N}\otimes\cdots\otimes V_{s_1}\otimes V(\Lambda_j)^{*a}\longrightarrow V(\Lambda_{i+1})\otimes V_{s_N}\otimes\cdots\otimes V_{s_1}\otimes V(\Lambda_{j+1})^{*a}\\
&T(z)=\Phi^*(z)\check{R}_{1,s_N}(z)\cdots\check{R}_{1,s_1}(z)\Phi(z).
\end{aligned}$$

The space $V(\Lambda_i)\otimes V_{s_N}\otimes\cdots\otimes V_{s_1}\otimes V(\Lambda_j)^{*a}$ is identified with $\mathcal{H}_{s_N\cdots s_1,ij}$ by the map $\Phi^{s_N,\cdots,s_1}(1,\cdots,1)$ and its inverse. Let us calculate the map $\tilde{T}(z)$ for which

$$\begin{array}{ccc}
\mathcal{H}_{s_N\cdots s_1,ij} & \xrightarrow{\Phi^{s_N,\cdots,s_1}(1,\cdots,1)} & V(\Lambda_i)\otimes V_{s_N}\otimes\cdots\otimes V_{s_1}\otimes V(\Lambda_j)^{*a}\\
\downarrow\tilde{T}(z) & & \downarrow T(z)\\
\mathcal{H}_{s_N\cdots s_1,i+1j+1} & \xrightarrow{\Phi^{s_N,\cdots,s_1}(1,\cdots,1)} & V(\Lambda_{i+1})\otimes V_{s_N}\otimes\cdots\otimes V_{s_1}\otimes V(\Lambda_{j+1})^{*a}
\end{array}$$

is a commutative diagram. Using the commutation relations 32 we have

$$\begin{aligned}
\tilde{T}(z) &= \Phi^{s_N,\cdots,s_1}(1,\cdots,1)^{-1}T(z)\Phi^{s_N,\cdots,s_1}(1,\cdots,1) \\
&= \Phi^{s_N,\cdots,s_1}(1,\cdots,1)^{-1}\Phi^*(z)\check{R}_{1,s_N}(z)\cdots\check{R}_{1,s_1}(z)\Phi(z)\Phi^{s_N,\cdots,s_1}(1,\cdots,1) \\
&= A_{i,N}(z)\Phi^*(z)\Phi^{s_N,\cdots,s_1}(1,\cdots,1)^{-1}\Phi^{s_N,\cdots,s_1}(1,\cdots,1)\Phi(z) \\
&= A_{i,N}(z)(1\otimes T_{XXZ}(z)),
\end{aligned}$$

where

$$A_{i,N}(z) = (-1)^{N+\sum_{j=1}^N s_j} q^{3\sum_{j=1}^N [\frac{s_j-\overline{i+j-1}}{2}] - \sum_{j=1}^N s_j + N + [\frac{N+i}{2}] - \sum_{j=1}^N s_j \overline{i+j-1}}.$$

Hence, up to a scalar factor, the transfer matrix coinsides with $1\otimes T_{XXZ}(z)$. If we normalize the eigenvalue of the vacuum vectors is equal to one, then the transfer matrix is given by $1\otimes T_{XXZ}(z)$.

9 Discussion

In this paper we introduce new kinds of q-vertex operators and using them propose the mathematical model of the inhomogeneous vertex models of the 6-vertex type. One of our vertex operators ${}_{V_n}\Phi^{V_{n+1}}(z)$ already appeared in Miki's paper[18] in the simplest non-trivial form $n=1$ in a different context.

It follows from our mathematical setting of the models that the excitation energies over the ground states are the same as that of the 6-vertex model. In our approach the impurity contributions to the several physical quantities may be calculated through the correlation functions.

As in the case of the other solvable lattice models[4, 10] the trace of the compositions of the new vertex operators satisfy certain q-difference equations. Except the case of the form $tr_{V(\Lambda_i)}(q^{-2\rho}\Phi(z_1)\cdots\Phi(z_k){}_{V_{s-1}}\Phi^{V_s}(z))$, those equations are different from the q-KZ equation with mixed spins. Hence the situation is rather unexpected from the point of view by the rough pictorial arguments[10, 4].

The new vertex operators can be considered as non-local operators actiong on the physical space of the XXZ-model. This fact may open the door to study the fusion model[19, 17] of the 6vertex model using the vertex operators defined here.

Obviously we can introduce the inhomogeneities in the spectral parameter (or the rapidities). This corresponds to consider the space $(V_{s_N-1})_{z_N}\otimes\cdots\otimes(V_{s_1-1})_{z_1}\otimes V(\Lambda_i)\otimes V(\Lambda_j)^{*a}$ etc.

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A Appendix1

In this section we give the integral formula for the matrix element $<\Lambda_{i+1}|O(\mathbf{z}|\mathbf{u})|\Lambda_i>$.

$$
\begin{aligned}
&<\Lambda_{i+1}|O(\mathbf{z}|\mathbf{u})|\Lambda_i> \\
&= \frac{1}{f} <\Lambda_{i+1}|\Phi_{\epsilon_1}(z_1)\cdots\Phi_{\epsilon_{n+1}}(z_n)\Psi_{\mu_n}(u_n)\cdots\Psi_{\mu_1}(u_1)|\Lambda_i> \\
&= (-1)^{s_2}(q-q^{-1})^{r_1+s_2}(-q)^{-i[\frac{n}{2}]+(i-1)[\frac{n+1}{2}]}\prod_{j:\mathrm{odd}}^{n+1}(-q^3z_j)^i\prod_{j:\mathrm{even}}^{n+1}(-q^3z_j)^{\frac{1}{2}}\prod_{j=1}^{n+1}(-q^3z_j)^{\frac{s_2-s_1}{2}} \\
&\prod_{k:\mathrm{even}}^{n}(-qu_k)^{\frac{1}{2}-i}\prod_a(q^2z_a)^{-1}\prod_b\prod_{j<b}(-q^3z_j)^{\frac{1}{2}}\prod_a\prod_{j<a}(-q^3z_j)^{-\frac{1}{2}}\prod_c\prod_{j>c}(-qu_j)^{\frac{1}{2}}\prod_d\prod_{j>d}(-qu_j)^{-\frac{1}{2}} \\
&\prod_a\int_{C_a}\frac{dw_a}{2\pi i}\prod_d\int_{C_d}\frac{d\xi_d}{2\pi i}\prod_a w_a^{-i+s_1-s_2}\prod_{a<b}w_a^{-1}\prod_{a<a'}w_a\prod_d\xi_d^{i-1}\prod_{d>c}\xi_d^{-1}\prod_{d>d'}\xi_d \\
&\frac{\prod_d\prod_{j=1}^{n+1}(1-\frac{\xi_d}{q^3z_j})\prod_a\prod_{l=1}^{n}(1-\frac{qu_l}{w_a})}{\prod_a\prod_{j\le a}(1-\frac{w_a}{q^2z_j})\prod_a\prod_{j\ge a}(1-\frac{q^4z_j}{w_a})\prod_d\prod_{k\le d}(1-\frac{u_k}{\xi_d})\prod_d\prod_{k\ge d}(1-\frac{\xi_d}{q^2u_k})} \\
&\frac{\prod_{a<a'}(1-\frac{w_{a'}}{w_a})(1-\frac{q^2w_{a'}}{w_a})\prod_{d<d'}(1-\frac{\xi_{d'}}{\xi_d})(1-\frac{\xi_{d'}}{q^2\xi_d})}{\prod_{a,d}(1-\frac{q\xi_d}{w_a})(1-\frac{\xi_d}{qw_a})}.
\end{aligned}
\tag{33}
$$

Here $r_1, r_2, s_1, s_2, a, b, c, d$ is defined as follows.

$$
\begin{aligned}
&\{a\}=\{j|\epsilon_j=0\},\quad \{b\}=\{j|\epsilon_j=1\},\quad \{c\}=\{j|\mu_j=0\},\quad \{d\}=\{j|\mu_j=1\},\\
&r_1=\sharp\{a\},\quad r_2=\sharp\{b\},\quad s_1=\sharp\{c\},\quad s_2=\sharp\{d\}.
\end{aligned}
$$

w_a and ξ_d are the integral variables. The integral contour C_a and C_d are taken in the following manner.

$$
\begin{aligned}
C_a &: \quad q^4z_j(j\ge a) \text{ and } q^{\pm1}\xi_d(\text{all } d) \text{ are inside,}\\
&: \quad q^2z_j(j\le a) \text{ are outside.}\\
C_d &: \quad u_k(k\le d) \text{ are inside,}\\
&: \quad q^2u_k(k\ge d) \text{ and } q^{\pm1}w_a(\text{all } a) \text{ are outside.}
\end{aligned}
$$

The special components are given by

$$
\begin{aligned}
&<\Lambda_1|O(\mathbf{z}|\mathbf{u})_{1\cdots1,0\cdots0}|\Lambda_0>= \\
&(-q)^{-[\frac{n+1}{2}]}\prod_{j:even}(-q^3z_j)^{\frac{1}{2}}\prod_{j=1}^{n+1}(-q^3z_j)^{\frac{1-j}{2}}\prod_{k:even}(-qu_k)^{\frac{1}{2}}\prod_{k=1}^{n}(-qu_k)^{\frac{k-1}{2}},
\end{aligned}
\tag{34}
$$

$$
\begin{aligned}
&<\Lambda_0|O(\mathbf{z}|\mathbf{u})_{0\cdots0,1\cdots1}|\Lambda_1>= \\
&(-q)^{-[\frac{n}{2}]+\frac{n(n+1)}{2}}\prod_{j:odd}(-q^3z_j)^{\frac{1}{2}}\prod_{j=1}^{n+1}(-q^3z_j)^{-\frac{j}{2}}\prod_{k:odd}(-qu_k)^{\frac{1}{2}}\prod_{k=1}^{n}(-qu_k)^{1-\frac{k}{2}}.
\end{aligned}
\tag{35}
$$

B Appendix2

We give the description of the level one vertex operators $\Phi(z)$ and $\Psi(z)$ on the free field realization of the represenattions[8].

$$\Phi_1(z) = \exp\sum_{n=1}^{\infty}(\frac{a_{-n}}{[2n]}q^{\frac{7n}{2}}z^n)\exp\sum_{n=1}^{\infty}(-\frac{a_n}{[2n]}q^{-\frac{5n}{2}}z^{-n})e^{\frac{\alpha}{2}}(-q^3z)^{\frac{\partial_\alpha+i}{2}},$$

$$\Phi_0(z,w) = \frac{(q-q^{-1})(q^2z)^{-1}}{(1-\frac{w}{q^2z})(1-\frac{q^4z}{w})}\exp\sum_{n=1}^{\infty}(\frac{a_{-n}}{[2n]}q^{\frac{7n}{2}}z^n - \frac{a_{-n}}{[n]}q^{\frac{n}{2}}w^n)$$

$$\exp\sum_{n=1}^{\infty}(-\frac{a_n}{[2n]}q^{-\frac{5n}{2}}z^{-n}+\frac{a_n}{[n]}q^{\frac{n}{2}}w^{-n})e^{-\frac{\alpha}{2}}w^{-\partial_\alpha}(-q^3z)^{\frac{\partial_\alpha+i}{2}},$$

$$\Psi_0(u) = \exp\sum_{n=1}^{\infty}(-\frac{a_{-n}}{[2n]}q^{\frac{n}{2}}u^n)\exp\sum_{n=1}^{\infty}(\frac{a_n}{[2n]}q^{-\frac{3n}{2}}u^{-n})e^{-\frac{\alpha}{2}}(-qu)^{\frac{-\partial_\alpha+i}{2}}(-q)^{-1+i},$$

$$\Psi_1(u,\xi) = \frac{-(q-q^{-1})\xi^{-1}}{(1-\frac{u}{\xi})(1-\frac{\xi}{q^2u})}\exp\sum_{n=1}^{\infty}(-\frac{a_{-n}}{[2n]}q^{\frac{n}{2}}u^n+\frac{a_{-n}}{[n]}q^{-\frac{n}{2}}\xi^n)$$

$$\exp\sum_{n=1}^{\infty}(\frac{a_n}{[2n]}q^{-\frac{3n}{2}}u^{-n}-\frac{a_n}{[n]}q^{-\frac{n}{2}}\xi^{-n})e^{\frac{\alpha}{2}}\xi^{\partial_\alpha}(-qu)^{\frac{-\partial_\alpha+i}{2}}(-q)^{-1+i},$$

$$\Phi_0(z) = \int_{C_1}\frac{dw}{2\pi i}\Phi_0(z,w),$$

$$\Psi_1(u) = \int_{C_2}\frac{d\xi}{2\pi i}\Psi_1(u,\xi),$$

where the contour C_1 and C_2 are specified by

$$C_1 : q^4z \text{ is inside and } q^2z \text{ is outside},$$
$$C_2 : u \text{ is inside and } q^2u \text{ is outside}.$$

C Appendix3

Here we give the OPE of the level one vertex operators. Notations are the same as that in [8] except that the normal orderings are carried out for $e^{n\alpha}$ and ∂_α.

$$\Phi_1(z_1)\Phi_1(z_2) = \gamma(\frac{z_1}{z_2})(-q^3z_1)^{\frac{1}{2}} : \Phi_1(z_1)\Phi_1(z_2) :,$$

$$\Phi_1(z_1)\Phi_0(z_2,w) = \gamma(\frac{z_1}{z_2})\frac{(-q^3z_1)^{-\frac{1}{2}}}{1-\frac{w}{q^2z_1}} : \Phi_1(z_1)\Phi_1(z_2,w) :,$$

$$\Phi_0(z_1,w)\Phi_1(z_2) = \gamma(\frac{z_1}{z_2})\frac{w^{-1}(-q^3z_1)^{\frac{1}{2}}}{1-\frac{q^4z_2}{w}} : \Phi_0(z_1,w)\Phi_1(z_2) :,$$

$$\Phi_0(z_1,w_1)\Phi_0(z_2,w_2) = \gamma(\frac{z_1}{z_2})w_1(-q^3z_1)^{-\frac{1}{2}}\frac{(1-\frac{w_2}{w_1})(1-\frac{q^2w_2}{w_1})}{(1-\frac{w_2}{q^2z_1})(1-\frac{q^4z_2}{w_1})} : \Phi_0(z_1,w_1)\Phi_0(z_2,w_2) :,$$

$$\Psi_0(u_1)\Psi_0(u_2) = \beta(\frac{u_1}{u_2})(-qu_1)^{\frac{1}{2}} : \Psi_0(u_1)\Psi_0(u_2) :,$$

$$\Psi_0(u_1)\Psi_1(u_2,\xi) = \beta(\frac{u_1}{u_2})\frac{(-qu_1)^{-\frac{1}{2}}}{1-\frac{\xi}{q^2u_1}} : \Psi_0(u_1)\Psi_1(u_2,\xi) :,$$

$$\Psi_1(u_1,\xi)\Psi_0(u_2) = \beta(\frac{u_1}{u_2})\frac{\xi^{-1}(-qu_1)^{\frac{1}{2}}}{1-\frac{u_2}{\xi}} : \Psi_1(u_1,\xi)\Psi_0(u_2) :,$$

$$\Psi_1(u_1,\xi_1)\Psi_1(u_2,\xi_2) = \beta(\frac{u_1}{u_2})\xi_1(-qu_1)^{-\frac{1}{2}}\frac{(1-\frac{\xi_2}{\xi_1})(1-\frac{\xi_2}{q^2\xi_1})}{(1-\frac{\xi_2}{q^2u_1})(1-\frac{u_2}{\xi_1})} : \Psi_1(u_1,\xi_1)\Psi_1(u_2,\xi_2) :,$$

$$\Phi_1(z)\Psi_0(u) = \alpha(\frac{z}{u})(-q^3z)^{-\frac{1}{2}} : \Phi_1(z)\Psi_0(u) :,$$

$$\Phi_1(z)\Psi_1(u,\xi) = \alpha(\frac{z}{u})(-q^3z)^{\frac{1}{2}}(1-\frac{\xi}{q^3z}) : \Phi_1(z)\Psi_1(u,\xi) :,$$

$$\Phi_0(z,w)\Psi_0(u) = \alpha(\frac{z}{u})w(-q^3z)^{-\frac{1}{2}}(1-\frac{qu}{w}) : \Phi_0(z,w)\Psi_0(u) :,$$

$$\Phi_0(z,w)\Psi_1(u,\xi) = \alpha(\frac{z}{u})w^{-1}(-q^3z)^{\frac{1}{2}}\frac{(1-\frac{qu}{w})(1-\frac{\xi}{q^3z})}{(1-\frac{q\xi}{w})(1-\frac{\xi}{qw})} : \Phi_0(z,w)\Psi_1(u,\xi) :,$$

$$\Psi_0(u)\Phi_1(z) = \omega(\frac{u}{z})(-qu)^{-\frac{1}{2}} : \Psi_0(u)\Phi_1(z) :,$$

$$\Psi_0(u)\Phi_0(z,w) = \omega(\frac{u}{z})(-qu)^{\frac{1}{2}}(1-\frac{w}{qu}) : \Psi_0(u)\Phi_0(z,w) :,$$

$$\Psi_1(u,\xi)\Phi_1(z) = \omega(\frac{u}{z})\xi(-qu)^{-\frac{1}{2}}(1-\frac{q^3z}{\xi}) : \Psi_1(u,\xi)\Phi_1(z) :,$$

$$\Psi_1(u,\xi)\Phi_0(z,w) = \omega(\frac{u}{z})\xi^{-1}(-qu)^{\frac{1}{2}}\frac{(1-\frac{q^3z}{\xi})(1-\frac{w}{qu})}{(1-\frac{qw}{\xi})(1-\frac{w}{q\xi})} : \Psi_1(u,\xi)\Phi_0(z,w) : .$$

Here

$$\gamma(z) = \frac{(q^2z^{-1})_\infty}{(q^4z^{-1})_\infty}, \quad \beta(z) = \frac{(z^{-1})_\infty}{(q^2z^{-1})_\infty}, \quad \alpha(z) = \frac{(qz^{-1})_\infty}{(q^{-1}z^{-1})_\infty}, \quad \omega(z) = \frac{(q^5z^{-1})_\infty}{(q^3z^{-1})_\infty}.$$

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